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ON THE METAPLECTIC ANALOG OF KAZHDAN'S "ENDOSCOPIC" LIFTING

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ABSTRACT
A simple "twisted" form of the Selberg trace formula, due to Kazhdan, is used to prove a metaplectic analog of Kazhdan's lifting of grossencharakters of a cyclic extension of degree r to automorphic representations of $GL(r)$.

Introduction

Generalizing classical results of Hecke and Maass, Shalika-Tanaka used the Weil representation to construct a correspondence between automorphic forms on $GL(1, \mathbf{A}_L)$ (i.e., grossencharakters of L) and certain automorphic forms on $SL(2, \mathbf{A}_K)$, where L/K is a separable quadratic extension of global fields (see, for example, [Gel2, Theorem 7.11]). If T denotes the Cartan subgroup of $SL(2)/\mathbf{A}_K$ associated to L, then this may be regarded as an instance of an "endoscopic" lifting from automorphic forms on $T(\mathbf{A}_K)$ to automorphic forms on $SL(2, \mathbf{A}_K)$ (see [LL]). In [Kaz], Kazhdan constructed the analogs of such liftings for $SL(r)$, T denoting the Cartan associated to a cyclic extension L/K of degree r. This was recently generalized further by Arthur–Clozel in [AC]. The object of this paper is to consider the metaplectic analog of Kazhadan's lifting, where $SL(r, \mathbf{A}_K)$ is replaced by its n-fold metaplectic covering group $\overline{SL(r, \mathbf{A}_K)}$ and $T(\mathbf{A}_K)$ by the n-fold cover $\overline{T(\mathbf{A}_K)}$. However, for various technical reasons, our results are somewhat less than hoped for; in particular, for the main result (3.40) we assume that r and n are relatively prime, that n is relatively prime to the least common multiple of all composite numbers less than or equal to r, and that K is totally imaginary. Thanks to the

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work of G. Prasad and M. S. Raganathan, it appears that the assumption $(r, n) = 1$ may be weakened considerably (and perhaps even dropped altogether). As in [Hen], we primarily work with $\overline{\mathrm{GL}(r)}$ rather than $\overline{\mathrm{SL}(r)}$, as one may pass from (local or global) representations of $\overline{\mathrm{GL}(r)}$ to (local or global) L-packets of representations of $\overline{\mathrm{SL}(r)}$ via the arguments in [LL], [GPS2, §1].

The contents of this note are as follows. In section one, the metaplectic analogs of the local Langlands, L-group, and Satake correspondences for GL(r) are explicitly worked out. In section two, some preliminary orbital integral identities are proven, and in section three, the main argument, using a simple case of the metaplectic Selberg trace formula, is given. At the referee's request I've added some remarks on the preservation of L-factors and ε-factors at the end of section 3. Further details are given in the introduction to each section.

Let me say that the paper of Cogdell and Piatetski-Shapiro [CPS] was a strong source of motivation for this work. There they constructed counterexamples to the metaplectic analog of the Langlands functoriality conjecture (in the case of quadratic base change) and the Howe duality conjecture (associated to the dual-reductive pair ($\overline{\mathrm{SL}(2)}$, PGL(2))). Whether or not the representations of $\overline{\mathrm{SL}(r, \mathbf{A}_K)}$ obtained by the method below have such interesting properties is yet to be seen. Because of the work of Shalika-Tanaka in the non-metaplectic case, one might wonder if they might turn out to be "theta" or "distinguished" representations associated to the classical theta functions of Kubota or those of Bump–Hoffstein [BH], but this turns out not to be the case. At any rate, in order to investigate base change for $\overline{\mathrm{SL}(2)}$ via the trace formula, certain (unproven) character identities are needed to effect the stabilization of the trace formula (see [LL, (2.4)] and subsection 3.3 below; this stabilization is a necessary prerequiste to any application of the trace formula). The search for a proof of these identities formed the initial motivation for this note.

Section 1

Metaplectic Langlands, L-group, and Satake Correspondences

For the metaplectic analog of the local Langlands' correspondence for GL(r), it seems that you have to replace r-dimensional representations (i.e., admissible homomorphisms) of the Weil–Deligne group by a certain class of *projective* representations (whose cocyles have values in the nth roots of unity), and replace semi-simple conjugacy classes in ${}^LG^0$ by semi-simple conjugacy classes of *nth powers* in ${}^LG^0$. Let me describe this in more detail.

We are given a non-archimedian local field F containing the nth roots of unity μ_n, char $F = 0$, always assuming that the residual characteristic p of F does not divide n. There is a metaplectic extension,

$$1 \to \mu_n \xrightarrow{i} \overline{\mathrm{GL}(r, F)} \xrightarrow{p} \mathrm{GL}(r, F) \to 1, \tag{1.1}$$

which splits over $K := \mathrm{GL}(r, \mathcal{O}_F)$.

FACT (1.2). (a) If $\gamma \in \mathrm{GL}(r, F)$ is regular then the centralizer $T(F) := \mathrm{Cent}(\gamma, \mathrm{GL}(r, F))$ is a torus.

(b) If $(\gamma, 1)^n \in \overline{\mathrm{GL}(r, F)}$ is regular (i.e., $\gamma^n \in \mathrm{GL}(r, F)$ is regular) then

$$\mathrm{Cent}((\gamma, 1)^n, \overline{\mathrm{GL}(r, F)}) = \overline{\mathrm{Cent}(\gamma^n, \mathrm{GL}(r, F))}.$$

This is an easy exercise.

As a matter of terminology, we will call a *metaplectic Cartan subgroup* (CSG for short) of $\overline{\mathrm{GL}(r, F)}$ a lift $\overline{T(F)}$ of a CSG $T(F)$ of $\mathrm{GL}(r, F)$. Of course, for $n > 1$ these metaplectic CSGs will be non-abelian. Recall that the CSGs of $\mathrm{GL}(r, F)$ are in 1–1 correspondence with the degree r extensions E of F where $T(F) = E^\times$. This and the fact above yield a complete intrinsic description of the metaplectic CSGs.

This first section is concerned with what amounts to the harmonic analysis of $\overline{T(F)}$ and some relations to the representation theory of $\overline{\mathrm{GL}(r, F)}$.

§1.1. Metaplectic Satake correspondence for tori

Since metaplectic tori are non-abelian, their irreducible representations non-trivial on μ_n (i.e., "genuine") will not be one-dimensional. However, those irreducible representations which are trivial on μ_n are all one-dimensional. In fact, the sets

$$\{\text{unramified characters of } \overline{T(F)}\} \text{ and } \{\text{unramified characters of } T(F)\}$$

are equal, where $T(F) = E^\times$ is the CSG defined by an unramified extension E/F, $[E : F] = r$.

In the non-metaplectic case, it is well-known that unramified characters of $T(F) = E^\times$ are in one-to-one correspondence with σ-conjugacy classes in ${}^LT^0(\mathbf{C})$ (σ denotes a fixed generator of the Galois group of E over F, $\mathrm{Gal}(E/F)$), *and* in one-to-one correspondence with the set of characters of the Hecke algebra

(1.3) $\mathscr{H}(T) := \{f\colon T(F) \to \mathbf{C} \mid f \text{ is bi-invariant under } T(F) \cap K\}.$

This may be found, for example, in [Bor, §§7, 9]; recall K denotes here the standard maximal compact-open (hyperspecial) subgroup of $GL(r, F)$. Let me now state this correspondence more precisely and then give its metaplectic analog.

First, how do we compare

$$\{\text{unramified characters of } T(E)\},$$

(resp. {unramified irreducible genuine representations of $\overline{T(F)}$}) with

$$\{\text{characters of } \mathscr{H}(T)\},$$

(resp., {irreducible representations of $\mathscr{H}(\bar{T})_{\text{gen}}$}), where

(1.4) $\mathscr{H}(\bar{T})_{\text{gen}} := \{\text{genuine } \tilde{f}\colon \overline{T(F)} \to \mathbf{C} \mid \tilde{f} \text{ is bi-invariant under } \overline{T(F)} \cap \bar{K}\}$?

This is easy: Given a representation χ of $\overline{T(F)}$, we obtain a representation of the Hecke algebra of $\bar{T}$, $\mathscr{H}(\bar{T})_{\text{gen}}$, by defining

$$\chi(\tilde{f}) := \int_{\overline{T(F)}} \tilde{f}(t)\chi(t)\,dt, \tag{1.5}$$

for $\tilde{f} \in \mathscr{H}(\bar{T})_{\text{gen}}$; one may check that

$$\chi(\tilde{f} * \tilde{g}) = \chi(\tilde{f})\chi(\tilde{g}).$$

Conversely, given a representation ρ of $\mathscr{H}(\bar{T})_{\text{gen}}$, one may check that

$$\chi(t) = \chi_\rho(t) := \rho(\text{id}_t), \tag{1.6}$$

for $t \in \overline{T(F)}$, defines a representation of $\overline{T(F)}$; here $\text{id} \in \mathscr{H}(\bar{T})_{\text{gen}}$ is the identity and the subscript t denotes translation,

$$\tilde{f}_t(x) := \tilde{f}(tx);$$

one may check that

$$\text{id}_t * \text{id}_s = \text{id}_{ts}$$

and that

$$\chi_\rho(t)\chi_\rho(s) = \chi_\rho(ts).$$

This yields the desired correspondence between representations of $\overline{T(F)}$ and representations of $\mathcal{H}(\bar{T})_{\text{gen}}$.

§1.2. Metaplectic L-group correspondence for tori

Next, let me recall the L-group correspondence for unramified CSGs. Let $\sigma \in \text{Gal}(E/F)$ be a fixed generator; it acts on ${}^LT^0$ by cyclic permutation of its coordinates. Since

$$t^{-1}(s \rtimes \sigma)t = t^{-1}s\sigma(t) \rtimes \sigma,$$

for $s, t \in {}^LT^0(\mathbf{C})$, it easily follows that ${}^LT^0$-conjugacy in ${}^LT^0 \rtimes \{\sigma\}$ is equivalent to σ-conjugacy in ${}^LT^0$. The correspondence between

$$\{\sigma\text{-conjugacy classes of elements of } {}^LT^0(\mathbf{C})\}$$

and

$$\{\text{unramified characters of } T(F)\}$$

is given explicitly by sending

$$t = (t_1, t_2, \ldots, t_r) \in {}^LT^0(\mathbf{C}) = (\mathbf{C}^\times)^r$$

to the character $\chi = \chi_t : E^\times \to \mathbf{C}^\times$ defined by

$$\chi(\pi_F) = t_1 \cdots t_r,$$

where π_F denotes a uniformizer for F. Since χ is unramified, this specifies it uniquely.

Now, what is the metapletic L-group correspondence for

$$\{\text{genuine unramified irreducible representations of } \overline{T(F)}\}?$$

First of all, we must know that all irreducible representations of $\overline{T(F)}$, non-trivial on μ_n, are obtained by induction from characters of the abelian subgroup $\overline{T(F)^n} = T(F)^n \times \mu_n$, where

$$T(F)^n = \{t^n \mid t \in T(F)\}$$

(similar to the situation in [Gel1, Theorem 5.1, p. 98]). This follows from what is sometimes called "Clifford–Mackey theory", so the proof of this will be omitted here (some general references are, for example, [Cl], [Karp], or [Mac]). Observe that the characters of $T(F)^n$ are precisely the nth powers of characters of $T(F) = E^\times$, so every genuine representation of $\overline{T(F)}$ must be induced from some character

$$(1.7) \qquad (t^n, \zeta) \mapsto \zeta \cdot \chi(t)^n \quad (\chi \text{ a character of } T(F))$$

of $\overline{T(F)^n} = T(F)^n \times \mu_n$. The set of these characters is in one-to-one correspondence with the set of nth powers of unramified characters χ of $T(F) = E^\times$. From the previous paragraph, it follows that there is a one-to-one (metaplectic L-group) correspondence between

$$(1.8) \qquad \{\text{genuine unramified irreducible representations of } \overline{T(F)}\}$$

and

$$(1.9) \qquad \{\sigma\text{-conjugacy classes of elements of } {}^L(T^n)^0(\mathbf{C})\},$$

where the *metaplectic L-group of $\overline{T(F)}$* is

$$(1.10) \qquad {}^L(T^n)^0(\mathbf{C}) = \{(z_1^n, \ldots, z_r^n) \mid z_i \in \mathbf{C}^\times\}.$$

This describes the L-group correspondence for unramified metaplectic CSGs $\overline{T(F)}$.

§1.3. Metaplectic L-group correspondence for $\overline{\mathrm{GL}(r)}$

What about the L-group correspondence for genuine unramified admissible representations of $\overline{\mathrm{GL}(r, F)}$? Let $\Pi_{\mathrm{unr}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}}$ denote the set of equivalence classes of such representations; if $n = 1$ then this is denoted $\Pi_{\mathrm{unr}}(\mathrm{GL}(r, F))$. The non-metaplectic L-group correspondence states that there is a canonical bijection between the sets

$$\{\text{semi-simple } \sigma\text{-conjugacy classes in } {}^LG^0 = \mathrm{GL}(r, \mathbf{C})\}$$

and $\Pi_{\mathrm{unr}}(\mathrm{GL}(r, F))$, given explicitly by sending

$$s = \{\sigma(g)^{-1} \cdot \mathrm{diag}(\chi_1(\pi), \ldots, \chi_r(\pi)) \cdot g \mid g \in {}^LG^0\},$$

for $\chi_i : F^\times \to \mathbf{C}^\times$ unramified, to the unique irreducible quotient of the parabolically induced representation

$$(1.11) \qquad \pi_s := \boxtimes_{i=1}^{r} \chi_i$$

(in the notation of Jacquet, Piatestski-Shapiro, Shalika; it is clear that each semi-simple class s of $\mathrm{GL}(r, \mathbf{C})$ defines an r-tuple $\chi_1, \ldots, \chi_r$ of unramified characters of $F^\times$ uniquely up to permutation). There is also the Satake correspondence between $\Pi_{\mathrm{unr}}(\mathrm{GL}(r, F))$ and characters of $\mathcal{H}(\mathrm{GL}(r, F))$, along with its metaplectic analog, which we will get to later. First, let me describe the metaplectic analog of the above L-group correspondence.

To be precise, in order to apply the results of [FK] one must either (a) assume the conjecture of [Kaz, p. 230], (b) assume that $(n, N) = 1$, where $N := \mathrm{l.c.m.}\{m \mid 1 \leqq m \leqq r, m \text{ composite}\}$, or (c) restrict ourselves to principal series representations rather than unramified representations. Assuming (a) or (b) from the metaplectic correspondence of Flicker–Kazhdan,

$$\Pi(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}} \to \Pi(\mathrm{GL}(r, F)), \tag{1.12}$$

we know that

$$\Pi_{\mathrm{unr}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}} \to \Pi_{\mathrm{unr}}(\mathrm{GL}(r, F)) \tag{1.13}$$

and that $\Pi_{\mathrm{unr}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}}$ consists entirely of principal series representations [FK, Introduction]. It is known that each $\bar{\pi} \in \Pi_{\mathrm{unr.p.s.}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}}$ is obtained by metaplectic parabolic induction from an unramified character $\chi = (\chi_1, \ldots, \chi_r)$ of

$$A_n(F) := \{a = \mathrm{diag}(a_1, \ldots, a_r) \mid a_i \in F^\times n \mid v_F(a_i) \text{ for all } i\}, \tag{1.14}$$

where $v_F : F^\times \to \mathbf{Z}$ denotes the usual normalized valuation, $v_F(\pi) = 1$. In fact, each principal series $\bar{\pi}$ is the irreducible quotient of some

$$\bar{\pi}_\chi = \mathrm{Ind}(\bar{B}, \overline{\mathrm{GL}(r, F)}, \bar{\rho}_\chi), \tag{1.15}$$

where

$$\bar{\rho}_\chi = \mathrm{Ind}(A_n(F) \times \mu_n, \overline{A(F)}, \chi), \tag{1.16}$$

extended to $\overline{B(F)}$ trivially. Here $A(F) \subset \mathrm{GL}(r, F)$ denotes the maximal split torus of diagonal matrices, and χ is defined on $\overline{A_n(F)} = A_n(F) \times \mu_n$ by

$$\chi : (a, \zeta) \mapsto \zeta \cdot \chi_1(a_1) \cdot \ldots \cdot \chi_r(a_r),$$

each χ_i being an unramified character of $F^\times$. In particular, we see that $\Pi_{\mathrm{unr.p.s.}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}}$ is in one-to-one correspondence with

$$\{\text{semi-simple } \sigma\text{-conjugacy classes in } {}^L(G^n)^0\},$$

where the *metaplectic L-group of* $\mathrm{GL}(r, F)$ is

$${}^L(G^n)^0 = \{g^n \mid g \in \mathrm{GL}(r, \mathbf{C})\}. \tag{1.17}$$

To be explicit, simply send the σ-conjugacy class of

$$\mathrm{diag}(\chi_1(\pi_F)^n, \ldots, \chi_r(\pi_F)^n)$$

to (when it is irreducible) $\bar{\pi}_\chi$, in the notation above. Under the metaplectic correspondence, this $\bar{\pi}_\chi$ will be sent to π_s, where s is the above σ-conjugacy class of $\mathrm{diag}(\chi_1(\pi_F)^n, \ldots, \chi_r(\pi_F)^n)$. This is the metaplectic L-group correspondence for genuine unramified (principal series) representations of $\overline{\mathrm{GL}(r, F)}$. By combining this correspondence with that of subsection 1.2, we obtain a map

$$\Pi_{\mathrm{unr.p.s.}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}} \to \Pi_{\mathrm{unr}}(\overline{T(F)})_{\mathrm{gen}}.$$

It should be remarked that if $(\bar{\pi}, V_{\bar{\pi}})$ is an unramified principal series, then $\bar{\pi}\mid_{\bar{K}}$ contains the identity representation exactly once, i.e., $\dim_{\mathbf{C}}(V_{\bar{\pi}}^{\bar{K}}) = 1$. This fact will be used to normalize a certain intertwining operator later on. For more facts on such representations, see [Gel1, §5.2], [FK].

§1.4. Metaplectic Satake correspondence for $\overline{\mathrm{GL}(r)}$

Now there is also, as mentioned, Satake's correspondence between unramified representations and characters of the Hecke algebra. In the non-metaplectic situation, this is described explicitly as follows. If $(\pi, V_\pi) \in \Pi(\mathrm{GL}(r, F))$, $f \in \mathscr{H}(\mathrm{GL}(r))$, where

$$\mathscr{H}(\mathrm{GL}(r)) := \{f\colon \mathrm{GL}(r, F) \to \mathbf{C} \mid f \text{ is compactly supported, locally constant, and bi-invariant under } K\}, \tag{1.18}$$

then

$$\pi(k)\pi(f)v = \pi(f)v,$$

for all $k \in K$, $v \in V_\pi$. In particular, if π is not unramified then $\mathrm{tr}\,\pi(f) = 0$, for $f \in \mathscr{H}(\mathrm{GL}(r))$. If π is unramified then it is of the form $\pi = \pi_s$, for some semi-simple $s \in \mathrm{GL}(r, \mathbf{C})$, and we define

$$S_0 f(s) := \mathrm{tr}\,\pi_s(f), \qquad f \in \mathscr{H}(\mathrm{GL}(r)). \tag{1.19}$$

Satake proved that, for each $f \in \mathscr{H}(\mathrm{GL}(r))$, $S_0(f)$ extends uniquely to a continuous function on ${}^LG^0 = \mathrm{GL}(r, \mathbf{C})$ such that

$$\begin{aligned} \mathscr{H}(\mathrm{GL}(r)) &\tilde{\to} \mathscr{A}_{\mathrm{cl.fcn.}}({}^LG^0), \\ f &\mapsto S_0(f), \end{aligned} \tag{1.20}$$

where $\mathscr{A}_{\text{cl.fcn.}}({}^LG^0)$ denotes the algebra (under point-wise multiplication) of regular class functions on the L-group of $\mathrm{GL}(r)$ (see [Bor, §§6, 7], [Car, §4], [Sat]). This map $f \mapsto S_0(f)$ actually defines a character of $\mathscr{H}(\mathrm{GL}(r))$; before proving this, let me state the metaplectic version. Let

$$\delta(a) := |\det \mathrm{Ad}(a)_{\mathrm{Lie}\, N(F)}|_F,$$

for $a \in A(F)$, where $N(F)$ denotes the unipotent radical of the Borel subgroup $B(F)$ of upper triangular matrices in $\mathrm{GL}(r, F)$, and define the *Satake transform* by

$$Sf(A) := \delta(a)^{1/2} \int_{N(F)} f(an)dn = \delta(a)^{-1/2} \int_{N(F)} f(na)dn, \tag{1.21}$$

for $\mathscr{H}(\mathrm{GL}(r))$. In the metaplectic case, define the metaplectic Satake transform by

$$\tilde{S}\tilde{f}(a) := \delta(p(a))^{1/2} \int_{N(F)} \tilde{f}(an)dn, \tag{1.22}$$

$a \in \overline{A(F)}$, $\tilde{f} \in \mathscr{H}(\overline{\mathrm{GL}(r)})_{\mathrm{gen}}$, where p is the surjection in the short exact sequence (1.1) defining $\overline{\mathrm{GL}(r)}$, and where $N(F)$ is identified with its image $N(F) \times \{1\}$ in $\overline{N(F)} = N(F) \times \mu_n$ in $\overline{\mathrm{GL}(r, F)}$; for $r = 2$, there is an algebra isomorphism

$$\psi : \mathscr{H}(\overline{\mathrm{GL}(r)})_{\mathrm{gen}} \xrightarrow{\sim} \mathscr{H}(\mathrm{GL}(r)),$$

such that

$$(\tilde{S}\tilde{f})(\tilde{a} \cdot \zeta_a^{-1}) = (S\psi(\tilde{f}))(a),$$

where

$$\zeta_\gamma := \beta(\gamma, \gamma)\beta(\gamma, \gamma^2) \cdots \beta(\gamma, \gamma^{n-1}),$$

$$\tilde{\gamma} := (\gamma^n, s(\gamma^n)^{-1}\zeta_\gamma) = (\gamma, 1)^n$$

(see [F]). In general, Flicker and Kazhdan [FK, §16] exhibit an explicit isomorphism between the Hecke algebra $\tilde{H}$ of $\overline{\mathrm{GL}(r, F)}$ with respect to its Iwahori subgroup $\tilde{I}$ and the Hecke algebra H of $\mathrm{GL}(r, F)$ with respect to its Iwahori subgroup I. Using this, they prove a bijection, independent of the trace formula, between unramified principal series of $\mathrm{GL}(r, F)$ and their metaplectic analogs (see [FK, Theorem in §16]). Using this and the results of [FK, §11], one can define a bijection, analogous to the one above, between

$\mathcal{H}(\overline{\mathrm{GL}(R)})_{\mathrm{gen}}$ and $\mathcal{H}(\mathrm{GL}(r))$. Consequently, the Satake transform $f \mapsto Sf$, resp. $\tilde{f} \mapsto \bar{S}\tilde{f}$, defines an algebra isomorphism

$$\mathcal{H}(\mathrm{GL}(r)) \xrightarrow{\sim} \mathcal{H}(A)^W, \tag{1.23}$$

resp.,

$$\mathcal{H}(\overline{\mathrm{GL}(r)})_{\mathrm{gen}} \xrightarrow{\sim} \mathcal{H}(\bar{A})^W_{\mathrm{gen}}, \tag{1.24}$$

where $W \cong S_r$ denotes the Weyl group of $\mathrm{SL}(r)$. To see that $S_0 f$ defines a character of $\mathcal{H}(\mathrm{GL}(r))$, recall that

$$S_0 f(s) = \int_{A(F)} Sf(a)\chi(a)da, \tag{1.25}$$

where

$$\chi(\mathrm{diag}(a_1, \ldots, a_r)) = \prod_{i=1}^{r} \chi_i(a_i),$$

and S corresponds to χ via (1.11) (see [Car, p. 149]). Since S is an algebra homomorphism, S_0 defines a character of the Hecke algebra. Similarly, the vector-valued transform

$$\bar{S}_0\tilde{f}(\chi) := \vec{\mathrm{tr}}\, \bar{\pi}_\chi(\tilde{f}) \quad (\vec{\mathrm{tr}} \text{ is defined below}), \tag{1.26}$$

for $\tilde{f} \in \mathcal{H}\overline{\mathrm{GL}(r)})_{\mathrm{gen}}$, χ a character of $\overline{A_r(F)}$, defines a representation of $\mathcal{H}(\overline{\mathrm{GL}(r)})_{\mathrm{gen}}$, since I *claim* that

$$\bar{S}_0\tilde{f}(\chi) = \int_{\overline{A(F)}} \bar{S}\tilde{f}(a)\bar{\rho}_\chi(a)da. \tag{1.27}$$

The verification of this uses no more than the (well-known) techniques of [Gel]: first, one computes the kernel of the operator $\bar{\pi}_\chi(\tilde{f})$, namely,

$$K_{\tilde{f},\chi}(k, k') = \int_{\mu_n \backslash \overline{B(F)}} \delta(b)^{1/2} \tilde{f}(k^{-1}bk')\bar{\rho}_\chi(b)db, \tag{1.28}$$

and, second, one computes the integral (which *defines* the vector-valued trace)

$$\vec{\mathrm{tr}}\, \bar{\pi}_\chi(\tilde{f}) = \int_K K_{\tilde{f},\chi}(k, k)dk, \tag{1.29}$$

where we have used the fact that $\bar{K} = K \times \mu_n$. As $p \nmid n$, we may identify K with its image $K \times \{1\}$ in $\bar{K}$. The "usual" trace,

$$\operatorname{tr} \bar{\pi}_\chi(\bar{f}) = \int_K \operatorname{tr} K_{\bar{f},\chi}(k, k)dk,$$

does not define an algebra homomorphism of the metaplectic Hecke algebra, so for our purposes the vector-valued trace is more useful.

§1.5. Metaplectic Langlands correspondence for $\overline{\mathrm{GL}(r)}$

Finally, let me describe the metaplectic Langlands correspondence between r-dimensional projective representations of the Weil group and admissible genuine representations of $\overline{\mathrm{GL}(r)}$, in the (simplest) case of unramified principal series representations.

Let me first recall the non-metaplectic situation. Let E/F be a cyclic degree r unramified extension and let ε be a character of $F^\times$ defining E via local class field theory ([Neu, pp. 41–44]), and let

$$r_E : W_E \to E^\times \tag{1.30}$$

denote the reciprocity map, where $W_E = E_{\bar{F}/E}$ denotes the Weil group of $\bar{F}/E$, normalized so that geometric Frobenius elements of W_E correspond to uniformizers of E ([Mor] describes the inverse of this map explicitly). The unramified Langlands correspondence is the following: take an unramified character χ of $E^\times$, so by induction from W_E to W_F the character $\chi \circ r_E$ yields an unramified r-dimensional representation of W_F. This representation corresponds to a unique unramified $\pi_s \in \Pi_{\mathrm{unr}}(\mathrm{GL}(r, F))$, and all such π_s arise in this way, where s is a conjugacy class as before. As we have already seen, the set of all such χ is in one-to-one correspondence with the set of σ-conjugacy classes in ${}^LT^0$; indeed, if $\chi(\pi_F) = \mu^r$, $\mu \in \mathbf{C}^\times$, then the character χ of $E^X = T(F)$ is associated to the σ-conjugacy class of

$$(\mu, \mu, \ldots, \mu) \times \sigma \in {}^LT^0 \times \sigma.$$

The representation π_s is that which is associated to the conjugacy class of $\mu \cdot P$, where $P = (P_{ij})$ is the permutation matrix given by

$$P_{ij} = \begin{cases} 1 & i \equiv j - 1 \bmod r, \\ 0 & \text{otherwise}; \end{cases} \tag{1.31}$$

if $\varepsilon(\pi_F) = \zeta$ then this is the conjugacy class of

$$\operatorname{diag}(\mu, \zeta\mu, \ldots, \zeta^{r-1}\mu) \in \mathrm{GL}(r, \mathbf{C}).$$

Furthermore, the central character of π_s satisfies

$$\omega_{\pi_s}(\pi_F) = (-1)^{r-1}\chi(\pi_F),$$

where π_F denotes a uniformizer for F as before. This may also be found in [Kaz, p. 223] or [Hen, §5.10].

Now for the metaplectic version. Let E/F be as above and let T be the CSG associated to E, so that $T(F) = E^{\times}$. One may view the metaplectic Langlands correspondence as a way of seeing how representations of $\overline{T(F)}$ yield elements of $\Pi_{\text{unr.p.s.}}(\overline{\mathrm{GL}(r, F)})_{\text{gen}}$. As mentioned before, Clifford–Mackey theory tells us that all the irreducible genuine representations of $\overline{T(F)}$ arise by induction from characters of $T(F)^n \times \mu_n$. As $\overline{T(F)}$ is a non-trivial extension of $E^{\times}$, each representation of $\overline{T(F)}$ may be regarded as a projective representation of $E^{\times}$, hence (by pulling back via the reciprocity map) as a projective representation of $W_E = W_{\bar{F}/E}$. This may be induced up to $W_F = W_{\bar{F}/F}$, to obtain a projective representation ρ whose cocycle has values in the nth roots of unity. It is this representation which is associated, via the unramified metaplectic Langlands correspondence, to a unique $\bar{\pi} = \bar{\pi}(\rho) \in \Pi_{\text{unr}}(\overline{\mathrm{GL}(r)})_{\text{gen}}$. This map, $\rho \mapsto \bar{\pi}(\rho)$, is described via L-parameters as in the non-metaplectic case: each genuine unramified irreducible (finite dimensional) representation of $\overline{T(F)}$ is associated to a unique σ-conjugacy class of nth powers,

$$(\mu^n, \dots, \mu^n) \in {}^L(T^n)^0,$$

as above (see §1.2), and each $\bar{\pi} \in \Pi_{\text{unr}}(\overline{\mathrm{GL}(r)})_{\text{gen}}$ is associated to a unique σ-conjugacy class

$$\mathrm{diag}(z_1^n, \dots, z_r^n),$$

in ${}^L(G^n)^0$, as above (see §1.3). Here the z_i are determined, up to an nth root of unity, by

$$z_i^n = \mu^n \cdot \zeta^{i-1}.$$

This completes our discussion of the metaplectic Langlands correspondence.

Section 2

ε-Orbital Integral Identities (Local Theory)

This section proves some identities needed in an argument in the next section. In the course of proving ε-orbital integral identities, it is necessary to investigate certain Hecke algebra homomorphisms which perhaps are of interest in their own right. In turn, these investigations yield some information on exactly which functions of t are those which can be expressed as an ε-orbital integral.

Beginning in subsection 2.6, we assume that n and r are relatively prime.

§2.1. The trace as an ε-orbital integral

Let me begin with some basic definitions, keeping the notation of the previous section. Let ε be a character of $F^\times$ of order r defining an unramified extension E/F via local class field theory; $\Gamma := \mathrm{Gal}(E/F) \cong \mathbf{Z}/r\mathbf{Z}$. Abusing notation, ε extends to $\mathrm{GL}(r, F)$ via $\varepsilon(g) := \varepsilon(\det g)$, so that it factors through $\mathrm{PGL}(r, F)$. For convenience, let

$$G := \mathrm{GL}(r), \quad G_\varepsilon(F) := \mathrm{Ker}(\varepsilon : G(F) \to \mathbf{C}^\times),$$

so that $Z(F) \cdot \mathrm{SL}(r, F) \subseteq G_\varepsilon(F)$. There is an extension (1.1) which, for $(p, n) = 1$, splits over K, and one may pull-back ε to $\overline{G(F)}$ via the surjection p in (1.1): $\varepsilon(\bar{g}) := \varepsilon(p(\bar{g}))$, for $\bar{g} \in \overline{G(F)}$; let $\overline{G_\varepsilon(F)} := \ker(\varepsilon : \overline{G(F)} \to \mathbf{C}^\times)$. For typographical simplicity, we shall sometimes write g in place of $\bar{g}$ when there is no chance of confusing elements of $G(F)$ with those of $\overline{G(F)}$. As a matter of definition, let us call an irreducible representation ω of $\overline{T(F)}$ *unramified* if it is induced from an unramified character of $T(F)^n$ as in (1.7). Let ω be an unramified genuine irreducible (finite dimensional) representation of the metaplectic CSG $\overline{T(F)}$, $T(F) = E^\times$, and let

$$\mathscr{H}(\bar{G}, \omega)_{\mathrm{gen}} \tag{2.1}$$

denote the convolution algebra of locally constant, genuine functions on $\overline{G(H)}$, bi-invariant under $\bar{K}$, transforming by ω^{-1} under $\overline{Z(F)^n}$, and compactly supported mod $\overline{Z(F)^n}$. Sometimes this will be simply called the *metaplectic Hecke algebra*.

For $\bar{f} \in \mathscr{H}(\bar{G}, \omega)_{\mathrm{gen}}$, $t \in \overline{T(F)}$, define the *ε-orbital integral* by

$$I(t, \bar{f}) := \int_{\overline{G(F)}/\overline{Z(F)^n}} \varepsilon(\det p(\bar{g}))\bar{f}(\bar{g}^{-1}t\bar{g})d\bar{g}, \tag{2.2}$$

where $d\bar{g}$ is a quotient measure on $\overline{G(F)}/\overline{Z(F)^n}$ to be specified later. Sometimes we shall write dg in place of $d\bar{g}$.

Following Kazhdan, we now introduce a certain element of the unramified principal series as follows. First, the non-metaplectic situation. From the L-

group correspondence for unramified principal series (see §1) it easily follows that there is, for each unramified character η of $T(F)=E^{\times}$, a $\pi_\eta \in \Pi_{\mathrm{unr}}(G(F))$ such that

(2.3a) π_η has central character η,

(2.3b) $\pi_\eta \cong \varepsilon \otimes \pi_\eta$.

In fact, there is only one unramified irreducible admissible representation satisfying (a), (b). The metaplectic L-group correspondence of the previous section yields a unique $\bar{\pi}_\omega \in \Pi_{\mathrm{unr.p.s.}}(\overline{G(F)})_{\mathrm{gen}}$ satisfying

(2.4a) $\bar{\pi}_\omega$ has central character $\omega \,|_{\overline{Z(F)^n}}$,

(2.4b) $\bar{\pi}_\omega \cong \varepsilon \otimes \bar{\pi}_\omega$.

(Uniqueness fails if the condition of being unramified is dropped.) It follows easily from [FK] that in order for $\bar{\pi}_\omega \mapsto \pi_\eta$, under the metaplectic correspondence, one must have $\eta(z)=\omega(z^n)$, $z \in Z(F)$.

It is remarkable, I think, that the ε-orbital integral above and this representation $\bar{\pi}_\omega$ are related by a simple identity, due to Kazhdan [Kaz] when $n=1$. It is this identity which is derived next. Indeed, it is shown that, roughly speaking, the ε-orbital integral $I(t,\bar{f})$ splits into the product of a function of t alone (independent of $\bar{f}$) and a "function" of $\bar{f}$ alone (namely, tr $\bar{\pi}_\omega(\bar{f})$).

In more detail, let $\sigma \in \Gamma$ be a fixed generator, pick an embedding $E^{\times} \to G(F)$ such that $\mathcal{O}_E^{\times} = E^{\times} \cap K$, and let

$$E_0 := \{t \in E \mid \sigma t = (-1)^{r-1} t\},$$

and

$$\tilde{\Delta}(t) := \prod_{1 \leq i < j \leq r} (\sigma^i t - \sigma^j t),$$

so that $\tilde{\Delta}(t) \in E_0$. From this, it is clear that $E_0 \cap \mathcal{O}_E^{\times}$ and that if $t, t' \in E_0 \cap E^{\times}$ then $t/t' \in F^{\times}$. Fixing a $t_0 \in E_0 \cap \mathcal{O}_E^{\times}$, we therefore have $E_0 = F \cdot t_0$. For $t \in E$, set

$$\Delta(t) := \varepsilon(\tilde{\Delta}(t) t_0^{-1}) \,|\tilde{\Delta}(t) t_0^{-1}|_F \,|N_{E/F} t|_F^{-(r-1)/2}; \tag{2.5}$$

this does not depend on the choice of t_0. Call $t \in E$ *regular* if $\tilde{\Delta}(t) \neq 0$; denote such elements by $E_{\mathrm{reg}} = (E^{\times})_{\mathrm{reg}} = T(F)_{\mathrm{reg}}$. This notion of regular is related to the "usual" notion of regular as follows: $t \in E^{\times}$ is regular if and only if it is regular semi-simple as an element of $G(F)$. In the non-metaplectic case, *Kazhdan's identity* states that, for $f \in \mathcal{H}(G,\eta)$, $t \in E_{\mathrm{reg}} \cap \mathcal{O}_E^{\times} = T(F)_{\mathrm{reg}} \cap K$, one has

$$I(t, f) = \Delta(t)^{-1} \operatorname{tr} \pi_\eta(f), \tag{2.6}$$

where the Haar measures are fixed as in [Hen, §5], [Kaz, p. 224].

The *metaplectic Kazhdan identity* is that, for $t \in \overline{T(F)}_{reg} \cap \bar{K}$, one has

$$I(t, \tilde{f}) = \Delta(p(t))^{-1} \operatorname{tr} \bar{\pi}_\omega(\tilde{f}), \tag{2.7}$$

for $\tilde{f} \in \mathscr{H}(\bar{G}, \omega)_{gen}$, where the Haar measure on $\overline{Z(F)^n}$ satisfies

$$\operatorname{vol}(\overline{Z(F)^n \cap K}) = \operatorname{vol}(Z(F) \cap K) \cdot [F^\times : (F^\times)^n]^{-1}, \tag{2.8}$$

and $\overline{G(F)}/\overline{Z(F)^n}$ gets the quotient measure.

§2.2. The proof of the metaplectic Kazhdan identity

Let me now prove the identity (2.7); the argument follows that in [Kaz, pp. 224–225] with a few modifications.

The proof requires some basic facts about spherical functions on the metaplectic group, so a small digression is necessary before beginning ([Car, §§3–4] assumes connectedness of his group; we require the metaplectic analog of [Car, pp. 149–150]). For each $\bar{\pi} \in \Pi_{unr.p.s.}(\overline{G(F)})_{gen}$ there is a χ as in Section 1.3 and a genuine unramified representation ω: $\overline{T(F)} \to \operatorname{Aut}(V_\omega)$ such that $\bar{\pi} = \bar{\pi}_\omega = \bar{\pi}_\chi$ (in the notation of (1.15), (2.24)). Set

$$\Phi_{\bar{K},\chi}(ank) := \bar{\rho}_\chi(a)\delta(a)^{1/2},$$

as in [Car, (33), p. 150], where $a \in \overline{A(F)}$, $r \in \overline{N(F)}$, $k \in \bar{K}$, and where $(\bar{\rho}_\chi, V_{\bar{\rho}_\chi})$ is as in Section 1.3, and set

$$\Gamma_{\bar{\pi}}(g) := \int_{\bar{K}} \Phi_{\bar{K},\chi}(kg)dk, \tag{2.9}$$

where $g \in \overline{G(F)}$. This might be referred to as the *vector-valued metaplectic spherical function*. It is easy to see that $\Gamma_{\bar{\pi}}(1) = 1 \in \operatorname{End}(V_{\bar{\rho}_\chi})$ and

$$\operatorname{tr} \Gamma_{\bar{\pi}} \in \mathscr{H}(\bar{G}, \bar{\rho}_\chi)_{gen} = \mathscr{H}(\bar{G}, \omega)_{gen}.$$

Note that for $k \in K$, $\det(k) \in \mathcal{O}_F^\times$ (for $k^r \in K$ but $v_F(\det(k^r)) \to \pm\infty$, and K is compact open). Consequently, since ε is unramified, $\varepsilon(\det(k)) = 1$ and

$$\varepsilon(ank) = \varepsilon(a),$$

for $a \in \overline{A(F)}$, $n \in \overline{N(F)}$, $k \in \bar{K}$. Since $\bar{\pi} = \bar{\pi}_\omega$ satisfies (2.4b), it is not hard to see that

$$\Phi_{\bar{K},\chi}(ank) = \varepsilon(a)\bar{\rho}_\chi(a)\delta\sigma(a)^{1/2},$$

and therefore

$$\Gamma_{\bar{\pi}}(g) = \varepsilon(g)\Gamma_{\bar{\pi}}(g), \qquad g \in \overline{G(F)}.$$

From this it is easy to see that

$$\text{(2.10)} \qquad \operatorname{supp}\Gamma_{\bar{\pi}} \subset \overline{G_\varepsilon(F)}.$$

Using the results of §1.4 (see also [Car, pp. 150–151]), one can show the following

FACT (2.11). For any $\bar{\pi} \in \Pi_{\mathrm{unr.p.s.}}(\overline{G(F)})_{\mathrm{gen}}$, $\bar{f} \in \mathscr{H}(\bar{G}, \omega)_{\mathrm{gen}}$, one has the identity

$$\overrightarrow{\operatorname{tr}}\,\bar{\pi}(\bar{f}) = \int_{\overline{G(F)}/\overline{Z(F)^n}} \bar{f}(x)\Gamma_{\bar{\pi}}(x)dx.$$

From these two facts (2.10–11) and the definition of $I(t, \bar{f})$, it is not hard to see that in order to prove our metaplectic Kazhdan identity, for $t \in \overline{T(F)}_{\mathrm{reg}} \cap \bar{K}$, one may assume without loss of generality that $\operatorname{supp}\bar{f} \subset \overline{G_\varepsilon(F)}$.

PROPOSITION (2.12) (Deligne–Kazhdan). *If* $\bar{f} \in \mathscr{H}(\bar{G}, \omega)_{\mathrm{gen}}$ *satisfies* $\operatorname{tr}\bar{\pi}(\bar{f}) = 0$, *for* $\bar{\pi} \in \Pi_{\mathrm{unr.p.s.}}(\overline{G(F)})_{\mathrm{gen}}$, *then* $I(t, \bar{f}) = 0$.

REMARK. For the proof in the connected case, see [DKV, §A.2], [Kaz, Proposition 1, pp. 224–225]. The connectedness assumption may be removed thanks to the work of Clozel [Cloz], and Vignéras [Vig2] (partially summarized in [KP2]).

From this proposition, it follows that

$$I(t, \bar{f}) = \phi_n(t)\operatorname{tr}\bar{\pi}(\bar{f}),$$

for some function ϕ_n on $\overline{T(F)}_{\mathrm{reg}} \cap \bar{K}$. By explicitly computing $\operatorname{tr}\bar{\pi}(\bar{f})$ and $I(t, \bar{f})$ for a certain $\bar{f}$, it is not too much trouble to show that

$$\phi_n(t) = \Delta(p(t))^{-1},$$

for a suitable choice of Haar measures. The remainder of this section is devoted to the details of this computation.

Define $\tilde{\theta}_\omega \in \mathscr{H}(\bar{G}, \omega)_{\mathrm{gen}}$ by

$$\tilde{\theta}_\omega(g) := \begin{cases} \omega(z)^{-1}, & g = kz, \quad z \in \overline{Z(F)^n}, \quad k \in \bar{K}, \\ 0, & \text{otherwise.} \end{cases}$$

From (2.11) and the fact that $\Gamma_{\bar{\pi}}$ is $\omega \mid_{\overline{Z(F)^n}}$ on the center, one obtains

$$\operatorname{tr} \bar{\pi}_\omega(\tilde{\theta}_\omega) = \dim(V_{\bar{\rho}_\chi}) = \deg(\bar{\rho}_\chi).$$

Also, since ω is unramified,

$$\begin{aligned} \int_{\overline{G(F)}/\overline{Z(F)^n}} \varepsilon(x)\tilde{\theta}_\omega(x^{-1}tx)dx &= \int_{\substack{\overline{G(F)}/\overline{Z(F)^n} \\ x^{-1}tx \in \bar{K}}} \varepsilon(x)dx \\ &= \int_{\substack{G(F)/Z(F) \\ x^{-1}tx \in \bar{K}}} \varepsilon(x)dx \\ &= \operatorname{vol}(Z(F)/Z(F)^n) \cdot \int_{\substack{G(F)/Z(F) \\ x^{-1}tx \in K}} \varepsilon(x)dx, \end{aligned}$$

for the Haar measures as in (2.8). Since $\operatorname{vol}(Z(F)/Z(F)^n)$ equals $[F^\times : (F^\times)^n]$, it follows from the calculation in [Kaz, pp. 225–227] that the last expression above is equal to

$$[F^\times : (F^\times)^n] \cdot \Delta(t)^{-1},$$

for $t \in T(F)_{\text{reg}} \cap K$. Putting all these together yields the metaplectic analog to Kazhdan's identity, as desired.

§2.3. The "second" metaplectic Kazhdan identity

As in [Kaz], [Hen, §5], fix the embedding $\iota : {}^LT \to {}^LG^0$ sending $(z_1, \ldots, z_r) \in {}^LT^0$ to $\operatorname{diag}(z_1, \ldots, z_r) \in {}^LG^0$ and $\sigma \in \Gamma$ to the permutation matrix $P = (P_{ij})$ in (1.31). From Section 1, we get a commutative diagram

(2.13)
$$\begin{array}{ccc} \mathscr{H}(G) & \xrightarrow{\iota_*} & \mathscr{H}(T) \\ \wr\downarrow & & \wr\downarrow \\ \mathscr{H}(\bar{G})_{\text{gen}} & \xrightarrow[\bar{\iota}_*]{} & \mathscr{H}(\bar{T})_{\text{gen}}. \end{array}$$

Let me first state, in the notation of this diagram, the non-metaplectic version of Kazhdan's "second" identity. It turns out to be equivalent to the first identity and is the following:

$$I(t,f)=\Delta(t)^{-1}\iota_*(f)(t), \tag{2.14}$$

for $t\in T(F)_{\mathrm{reg}}$ (not just $t\in T(F)_{\mathrm{reg}}\cap K$), and $f\in\mathscr{H}(G)$. Of more interest to us is the *metaplectic "second" identity*:

$$I(t,\tilde f)=\Delta(p(t))^{-1}\tilde\iota_*(\tilde f)(t), \tag{2.15}$$

for $t\in\overline{T(F)}_{\mathrm{reg}}$, $\tilde f\in\mathscr{H}(\bar G)_{\mathrm{gen}}$. Before proving this identity, one preliminary fact is needed.

Kazhdan's proof of the "second" identity ([Hen. p. 169]) gives

$$\eta(\iota_*(f))=\operatorname{tr}\pi_\eta(f),\qquad f\in\mathscr{H}(G), \tag{2.16}$$

where $\eta\in\{\text{unramified characters of } T(F)\}$ is regarded as a character of $\mathscr{H}(T)$ and $\pi_\eta\in\Pi_{\mathrm{unr}}(G(F))$ is the corresponding principal series representation. To prove the metaplectic analog, it is necessary to first prove (in the notation of §2.1)

$$(\operatorname{tr}\omega)(\tilde\iota_*(\tilde f))=\operatorname{tr}\tilde\pi_\omega(\tilde f), \tag{2.17}$$

for $\tilde f\in\mathscr{H}(\bar G)_{\mathrm{gen}}$, where ω is regarded as a finite-dimensional representation of $\mathscr{H}(\bar F)_{\mathrm{gen}}$ by

$$\omega:\tilde h\mapsto\int_{\overline{T(F)}}\omega(t)\tilde h(t)dt,$$

where $\tilde h\in\mathscr{H}(\bar T)_{\mathrm{gen}}$, and ω is as in (2.4). Note that (2.17) is more or less the "$\overline{T(F)}$-analog" of the identity (1.27) in §1.4. In fact, there is a commutative diagram

$$\begin{array}{ccc} \mathscr{H}(\bar G)_{\mathrm{gen}} & \xrightarrow{\tilde\iota_*} & \mathscr{H}(\bar T)_{\mathrm{gen}} \\ {\scriptstyle \tilde S_0}\downarrow & & \uparrow{\scriptstyle \psi_*} \\ \mathbf{C}^{\dim(\rho_\chi)} & \xrightarrow{\rho_\chi} & \mathscr{H}(\bar A)_{\mathrm{gen}}, \end{array} \tag{2.18}$$

for some ω as above associated to an unramified χ as in (1.15–16), where ψ_* is defined below via the L-group correspondence. This is actually not hard to verify using the description of all the irreducible representations of $\mathscr{H}(\bar T)_{\mathrm{gen}}$ given in §1.1. What was discovered there was that there is a character χ of $\overline{A(F)}^r$ such that

(2.19) $$\omega(\psi_*(\tilde{h})) = \tilde{\rho}_\chi(\tilde{h}),$$

for $\tilde{h} \in \mathscr{H}(\bar{A})_{\text{gen}}$, where

$$\psi_* : \mathscr{H}(\bar{A})_{\text{gen}} \to \mathscr{H}(\bar{T})_{\text{gen}}$$

is the homomorphism defined via the L-group correspondence (here $\psi : {}^L T^0 \times \Gamma \to {}^L A^0$ sends $(z_1, \ldots, z_r) \times \sigma$ to $(z_1, \zeta z_2, \ldots, \zeta^{r-1} z_r)$). By using L-parameters, it is not hard to verify that the diagram

(2.20) $$\begin{array}{ccc} \mathscr{H}(\bar{G})_{\text{gen}} & \xrightarrow{i^*} & \mathscr{H}(\bar{T})_{\text{gen}} \\ \bar{S} \downarrow & & \uparrow \psi_* \\ \mathscr{H}(\bar{A})_{\text{gen}} & = & \mathscr{H}(\bar{A})_{\text{gen}} \end{array}$$

commutes. These results, combined with the identity

(2.21) $$\vec{\text{tr}}\, \bar{\pi}_\chi(\bar{f}) = \bar{\rho}_\chi(\bar{S}\bar{f})$$

(see §1.4), yields the identity (2.17), as desired.

Using the identity (2.17) along with the computation in [Hen, §5.11, p. 169], one may now prove the metaplectic "second" identity (2.15). To extend Henniart's computation to the metaplectic case, only one small modification is needed: one uses the fact that every $t \in \overline{T(F)}_{\text{reg}}$ may be written in the form $t = t_0 \cdot (\pi^m, 1)$, for a unique $m \in \mathbf{Z}$, where $\pi \in F^\times$ denotes a fixed uniformizer and $t_0 \in \overline{T(F)}_{\text{reg}} \cap \bar{K}$ (the proof of this may be easily reduced to the non-metaplectic case, when it follows from the fact that E/F is unramified). For further details, I refer to [Hen, p. 169].

§2.4. The generalized metaplectic Kazhdan identity

A generalization of the "second" metaplectic identity is needed, so let me *change notation* as follows. Let $r = md$, E/F be an unramified extension (hence cyclic) of degree m, ε a character of $F^\times$ of order m defining E, and $\sigma_0 \in \Gamma := \text{Gal}(E/F)$ a fixed generator. Let

$$H(F) := \text{GL}_d(E),$$

$$K_H := \text{GL}_d(\mathcal{O}_E),$$

$$T(F) = T_H(F) := \text{diagonal subgroup of } \text{GL}_d(E),$$

and let S_H denote the Satake transform of $H(F)$, considered as the group $\mathrm{GL}_d(E)$. The dual group of H is the semi-direct product

$$^LH = {}^LH^0 \times \Gamma, \qquad {}^LH^0 = \mathrm{GL}_d(\mathbf{C})^m,$$

and σ_0 acts by $\sigma_0(x_1, \ldots, x_m) = (x_m, x_1, \ldots, x_{m-1})$. Associated to the natural embedding $^LH \to {}^LG$, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(G) & \xrightarrow{j_*} & \mathcal{H}(H) & \xrightarrow{S_H} & \mathcal{H}(T) \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \mathcal{H}(\tilde{G})_{\mathrm{gen}} & \xrightarrow[\tilde{j}_*]{} & \mathcal{H}(\tilde{H})_{\mathrm{gen}} & \xrightarrow[\tilde{S}_H]{} & \mathcal{H}(\tilde{T})_{\mathrm{gen}}. \end{array} \tag{2.23}$$

Let's change notation further by redefining the *ε-orbital integral* by

$$I(t,\tilde{f}) := \int_{\overline{G(F)}/\overline{T(F)}} \varepsilon(x)\tilde{f}(x^{-1}tx)dx, \tag{2.24}$$

for $\tilde{f} \in \mathcal{H}(\tilde{G})_{\mathrm{gen}}$, $t \in \overline{T(F)}_{\mathrm{gen}}$.

The following ε-orbital integral identity generalizes Kazhdan's second identity:

$$I(t,\tilde{f}) = \Delta(p(t))^{-1}(\tilde{S}_H \circ \tilde{j}(\tilde{f}))(t), \tag{2.25}$$

for $t \in \overline{T(F)}_{\mathrm{reg}}$, $\tilde{f} \in \mathcal{H}(\tilde{G})_{\mathrm{gen}}$, and for a suitable choice of Haar measures. This is what might be referred to as the *generalized second identity*.

The proof, see [Hen, p. 171], goes roughly as follows. Using $\overline{G(F)} = \overline{P(F)} \cdot \tilde{K}$, where $P(F)$ is a parabolic with Levi subgroup $M(F) = \mathrm{GL}_d(F)^m$ and having $t \in \overline{P(F)}$, it is easy to see that

$$I(t,\tilde{f}) = \int_{\overline{P(F)}/\overline{T(F)}} \varepsilon(p)\tilde{f}(p^{-1}tp)dp,$$

for $\tilde{f} \in \mathcal{H}(\tilde{G})_{\mathrm{gen}}$, $t \in \overline{T(F)}_{\mathrm{reg}}$, and Haar measures as in [Car]. Since $\overline{P(F)} = \overline{M(F)} \cdot \overline{N(F)}$, this is

$$= \int_{\overline{M(F)}/\overline{T(F)}} \int_{\overline{N(F)}} \varepsilon(mn)\tilde{f}(n^{-1}m^{-1}tnm)dmdn$$

$$= \int_{\overline{M(F)}/\overline{T(F)}} \varepsilon(m)F_{\tilde{f}}(m^{-1}tm)dm,$$

where

(2.26) $$F_{\tilde f}(m) := \int_{\overline{N(F)}} \tilde f(n^{-1}mn)\,dn = \Delta_N(p(m))^{-1} \int_{N(F)} \tilde f(\tilde n m)\,dn,$$

for $m \in \overline{M(F)}$, $\tilde n := (n, 1)$, p as in (1.1). Here of course

$$\Delta_N(m) := |\det(\mathrm{Ad}(m) - 1)_{\mathrm{Lie}\, N(F)}|_F,$$

by [Car, p. 145]. Setting

(2.27) $$I^{\bar A}(t, \tilde h) := \int_{\overline{M(F)}/\overline{T(F)}} \varepsilon(m) \tilde h(m^{-1}tm)\,dm,$$

for $\tilde h \in \mathscr{H}(\bar M)_{\mathrm{gen}}$, the above may be rewritten as

(2.28) $$I(t, \tilde f) = \Delta_N(p(t))^{-1} \delta(p(t))^{1/2} I^{\bar A}(t, \tilde S_{G/M} \tilde f),$$

where δ is as in §1.4 and

(2.29) $$\tilde S_{G/M} \tilde f(m) = \delta(p(m))^{-1/2} \int_{N(F)} \tilde f(\tilde n m)\,dn.$$

As in (2.23), there is the commutative diagram

(2.30) $$\begin{array}{ccc} \mathscr{H}(M) & \xrightarrow{(j_M)_*} & \mathscr{H}(T) \\ \wr\big\downarrow & & \wr\big\downarrow \\ \mathscr{H}(\bar M)_{\mathrm{gen}} & \xrightarrow{(\tilde j_{\bar M})_*} & \mathscr{H}(\bar T)_{\mathrm{gen}}, \end{array}$$

where $(j_M)_*$ is the Hecke algebra homomorphism denoted by $(\iota_A)_*$ in [Hen. pp. 172–173]. In the notation of this diagram, the "second" identity says that

(2.31) $$I^{\bar A}(t, \tilde h) = \Delta_M(p(t))^{-1} (\tilde j_{\bar M})_*(\tilde h)(t),$$

where Δ_M is as in [Hen, pp. 172–173]. The identity (2.25) now follows from (2.31), the relation

$$\Delta(t) = \Delta_N(t) \delta(t)^{-1/2} \Delta_M(t),$$

and the fact that

$$(\tilde j_{\bar M})_* \circ \tilde S_{G/M} = \tilde S_H \circ \tilde j_*.$$

The former identity is proven in [Hen, pp. 172–173] and the latter identity follows from the commutativity of the diagrams

$$\begin{array}{ccccc} \mathcal{H}(G) & \xrightarrow{S_{G/M}} & \mathcal{H}(M) & \xrightarrow{(j_M)_*} & \mathcal{H}(T) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathcal{H}(\bar{G})_{\mathrm{gen}} & \xrightarrow{\tilde{S}_{G/M}} & \mathcal{H}(\bar{M})_{\mathrm{gen}} & \xrightarrow{(\tilde{j}_{\bar{M}})_*} & \mathcal{H}(\bar{T})_{\mathrm{gen}} \end{array} \tag{2.33}$$

and

$$\begin{array}{ccccc} \mathcal{H}(G) & \xrightarrow{j_*} & \mathcal{H}(H) & \xrightarrow{S_H} & \mathcal{H}(T) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathcal{H}(\bar{G})_{\mathrm{gen}} & \xrightarrow{\tilde{j}_*} & \mathcal{H}(\bar{H})_{\mathrm{gen}} & \xrightarrow{\tilde{S}_H} & \mathcal{H}(\bar{T})_{\mathrm{gen}}, \end{array} \tag{2.34}$$

and the fact that (see [Hen, p. 172])

$$S = S_G = (j_M)_* \circ S_{G/M}, \quad S_G = S_H \circ j_*,$$

$$\tilde{S} = \tilde{S}_G = (\tilde{j}_{\bar{M}})_* \circ \tilde{S}_{G/M}, \quad \tilde{S}_G = \tilde{S}_H \circ \tilde{j}_*.$$

§2.5. A metaplectic "fundamental lemma"

Recall that the metaplectic "second" ε-orbital integral identity was derived from the relation (2.17). This relation turns out to be a special case of the "fundamental lemma" proven below. Our relation is, in fact, the analogous relation associated to the generalized second identity (2.25) and is of basic importance in an argument of section 3.

Let ω denote an unramified character of $\overline{Z(F)^n}$. Such a character, of course, arises from a character of $Z(F)^n$ by Clifford theory.

LEMMA (2.35) (*"metaplectic fundamental lemma for $\overline{T(F)}$"*).

(a) *For each $\tilde{f} \in \mathcal{H}(\bar{G}, \omega)_{\mathrm{gen}}$ there is a unique $\tilde{f}^T \in \mathcal{H}(\bar{T}, \omega\varepsilon^{r(r-1)/2})_{\mathrm{gen}}$ such that*

$$\tilde{f}^T(t) = \Delta(p(t))I(t, \tilde{f}),$$

for $t \in \overline{T(F)}_{\mathrm{gen}}$.

(b) *The map $\tilde{f} \mapsto \tilde{f}^T$ defines an algebra homomorphism*

$$\mathcal{H}(\bar{G}, \omega)_{\mathrm{gen}} \to \mathcal{H}(\bar{T}, \omega\varepsilon^{r(r-1)/2})_{\mathrm{gen}}.$$

(*For the image, see* (3.11) *in* §3 *below.*)

(c) *For any unramified irreducible genuine representation* μ *of* $\overline{T(F)}$ *such that* $\mu\,|_{\overline{Z(F)^n}} \equiv \omega$, *the following identity holds for all* $\bar{f} \in \mathscr{H}(\bar{G}, \omega)_{\text{gen}}$:

$$(\operatorname{tr}\mu)(\bar{f}^{\bar{T}}) = \operatorname{tr}\bar{\pi}_\mu(\bar{f}),$$

where $\bar{\pi}_\mu \in \Pi_{\text{unr}}\overline{(G(F))}_{\text{gen}}$ *is the element associated to* μ *as in* (2.4).

This is the metaplectic analog of [Hen, Proposition 5.17, p. 173]; its proof follows the non-metaplectic case fairly closely; but the argument, which is mostly a matter of piecing together results proven above, is sketched below for completeness.

Let me begin by stating a few simple facts. First of all, the map

$$\underset{\sim}{\omega}(\bar{f})(g) := \int_{\overline{Z(F)^n}} \bar{f}(gz)\omega(z)\,dz \tag{2.36}$$

defines a surjective algebra homomorphism

$$\underset{\sim}{\omega} : \mathscr{H}(\bar{G})_{\text{gen}} \to \mathscr{H}(\bar{G}, \omega)_{\text{gen}}.$$

Similarly, the map

$$\underset{\sim}{\omega}(\bar{h})(t) := \int_{\overline{Z(F)^n}} \bar{h}(tz)\omega(z)\varepsilon(z)^{r(r-1)/2}\,dz, \tag{2.37}$$

defines a surjective algebra homomorphism

$$\underset{\sim}{\omega} : \mathscr{H}(\bar{T})_{\text{gen}} \to \mathscr{H}(\bar{T}, \omega\varepsilon^{r(r-1)/2})_{\text{gen}}.$$

Thanks to the identity (2.25), one can verify that (a), (b) of the lemma hold with

$$(\underset{\sim}{\omega}(\bar{f}))^{\bar{T}} := \underset{\sim}{\omega}(\tilde{S}_H \circ \tilde{j}_*(\bar{f})), \tag{2.38}$$

for $\bar{f} \in \mathscr{H}(\bar{G})_{\text{gen}}$. For (c), observe that

$$\mu(\underset{\sim}{\omega}(\bar{h})) = \mu(\bar{h}), \tag{2.39}$$

for $\bar{h} \in \mathscr{H}(\bar{T})_{\text{gen}}$, and

$$\operatorname{tr}\bar{\pi}(\underset{\sim}{\omega}(\bar{f})) = \operatorname{tr}\bar{\pi}(\bar{f}), \tag{2.40}$$

for $\bar{f} \in \mathscr{H}(\bar{G})_{\text{gen}}$. Part (c) now follows from these, (2.17), and the diagrams (2.18), (2.23).

§2.6. Basic facts about metaplectic L-packets

Here certain aspects of metaplectic (local) L-indistinguishability will be considered. In particular, some ("twisted") character relations are derived between representations of $\overline{G(F)}$ and L-packets of representations of $\overline{G_\varepsilon(F)}$. Also, and more important for us, the analogs of these relations for metaplectic covers of the multiplicative group of a division algebra are given. The results of this subsection are basically some analogs of a simple observation of Kazhdan [Kaz, Lemma 1, p. 216].

Before beginning, let me introduce some notation. Let F be a p-adic field containing μ_n, as usual, with $(p, n) = 1$, let K be a number field containing μ_n, let $\Sigma(K)$ denote the set of places of K, $\Sigma_f(K)$ the set of finite places of K, and $\Sigma_\infty(K)$ the set of infinite (archimedian) places of K. Let me assume that K is totally imaginary. Although this is probably an unnecessary assumption, it implies that all our metaplectic extensions will split at the infinite places, so that there is no need to worry about extending the above local analysis from p-adic fields to archimedian fields. Let D be a central simple division algebra over K of degree r^2 which is non-split only at a finite set $S \subset \Sigma(K)$ (this *defines* S, so it deends on D/K), and such that, for all $v \in S$, D_v is a division algebra. For typographical simplicity, when $F = K_v$ let us write D/F or simply D in place of D_v when it is clear that we are working locally. Let

$$\varepsilon : \mathbf{A}_K^\times / K^\times \rightarrow \mathbf{C}^\times$$

denote a character of order r (hence unitary), and let L denote the cyclic extension of K defined by ε via global class field theory [Neu]. For each $v \in \Sigma(K)$, ε_v is a character of $K_v^\times$ which we sometimes simply denote by ε when it is clear that we are working locally. Let $\Gamma := \mathrm{Gal}(L/K) \cong \mathbf{Z}/r\mathbf{Z}$ and let

$$\Pi^\varepsilon(G(F)) := \{\pi \in \Pi(G(F)) \mid \pi \cong \varepsilon \otimes \pi\},$$

$$\Pi^\varepsilon(\overline{G(F)})_{\mathrm{gen}} := \{\bar\pi \in \Pi(\overline{G(F)})_{\mathrm{gen}} \mid \bar\pi \cong \varepsilon \otimes \bar\pi\},$$

if ε denotes a character of $F^\times$.

Let us assume that L ramifies "away from S", i.e. that for all $v \in S$, ε_v is of order (exactly) r. This implies that there is an embedding $L \hookrightarrow D$; if $v \notin S$, $F = K_v$, $E = L_w$, $w \mid v$ with v unramified in L, then $D_v^\times$ is isomorphic to $G(F)$, and we want to choose this isomorphism and the embedding $L \hookrightarrow D$ compatible with the local embedding $E^\times \hookrightarrow G(F)$ chosen just following (2.4) above. Let G_D denote the algebraic group defined over K by

$$G_D(R) := (D \otimes_K R)^\times,$$

for any K-algebra R.

Assume that $(n, r) = 1$ from now on. As in [KP2, §6], the group $G_D(\mathbf{A}_K)$ has an n-fold metaplectic extension $\overline{G_D(\mathbf{A}_K)}$,

$$1 \to \mu_n \xrightarrow{i} \overline{G_D(\mathbf{A}_K)} \xrightarrow{p} G_D(\mathbf{A}_K) \to 1, \tag{2.41}$$

which splits over $G_D(K)$. It is a consequence of a result of Klose [Kl] that we may choose D, in case D is a quaternion algebra, such that $\overline{G_D(\mathbf{A}_K)}$ exists without assuming $(n, r) = 1$. (I believe that it is a consequence of deep work of G. Prasad and M. S. Raganathan (not yet completely published, but see [PR1], [PR2], [P]) that even in the case when D has degree larger than 4, we can choose D such that a suitable topological covering $\overline{G_D(\mathbf{A}_K)}$ exists.) Moreover, if $v \notin S$ then $D_v^\times \cong G_D(K_v) \cong G(K_v)$ and there is an extension (2.41) such that

$$\begin{array}{ccccc} \mu_n & \xrightarrow{i_v} & \overline{G_D(K_v)} & \xrightarrow{p_v} & G_D(K_v) \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \mu_n & \longrightarrow & \overline{G(K_v)} & \longrightarrow & G(K_v). \end{array} \tag{2.42}$$

The global metplectic groups $\overline{G_D(\mathbf{A})}$, $\overline{G(\mathbf{A}_K)}$ are not restricted direct products of their local analogs, but they are subgroups of the restricted direct product which only differs by a central subgroup (see [KP1], [GPS1, §8]). The automorphic representations which will be considered on $\overline{G_D(\mathbf{A}_K)}$, $\overline{G(\mathbf{A}_K)}$ lift to factorizable representations on the restricted direct product. Later, in order to define the global "Hecke algebra" for $\overline{G_D(\mathbf{A}_K)}$, it will be necessary to choose a model of G_D defined over a certain ring of "S-integers", however (2.41) and (2.42) shall still hold true, of course. Let

$$\mathrm{Nrd} : D \to K$$

denote the reduced norm; the same symbol is used to denote its (unique) extension to D_v.

Before proving character relations, it is best to record some lemmas on some elementary aspects of metaplectic L-indistinguishability. Let $\varepsilon : F^\times \to \mathbf{C}^\times$ denote a character of order (exactly) r. Clifford–Mackey theory (see the argument in [Kaz, pp. 216–217]) gives us the following simple

FACT (2.43). (a) For each $\tilde{\pi} \in \Pi^\varepsilon(\overline{G(F)})_{\mathrm{gen}}$, $\tilde{\pi}\,|_{\overline{G_\varepsilon(F)}}$ decomposes into a direct sum of exactly n irreducible $\tilde{\rho}_i \in \Pi(\overline{G_\varepsilon(F)})_{\mathrm{gen}}$, $i = 1, \ldots, r$. Furthermore,

$\overline{G(F)}/\overline{G_\varepsilon(F)}$ acts transitively on the set $\{\bar{\rho}_i \mid 1 \leqq i \leqq r\}$, by $\bar{\rho}_i \mapsto \bar{\rho}_i^g$, where (as always) $\bar{\rho}^g(x) := \bar{\rho}(g^{-1}xg)$.

(b) For any $i = 1, \ldots, r$, $\bar{\pi} = \mathrm{Ind}(\overline{G_\varepsilon(F)}, \overline{G(F)}, \bar{\rho}_i)$.

Let

$$G_{D\varepsilon}(F) := \ker(\varepsilon \circ \mathrm{Nrd} : G_D(F) \to \mathbf{C}^\times),$$

and similarly for $\overline{G_{D_\varepsilon}(F)}$. The division algebra analog of (2.43) is the following

FACT (2.44). (a) For each $\bar{\pi} \in \Pi^\varepsilon(\overline{G_D(F)})_{\mathrm{gen}}$, $\bar{\pi}\,|_{\overline{G_{D\varepsilon}(F)}}$ decomposes into a direct sum of exactly r irreducibles $\bar{\rho}_i \in \Pi(\overline{G_{D\varepsilon}(F)})_{\mathrm{gen}}$, $i = 1, \ldots, r$, and $\overline{G_D(F)}/\overline{G_{D\varepsilon}(F)}$ acts transitively on the set $\{\bar{\rho}_i\}$, by conjugation.

(b) For each $i = 1, \ldots, r$, $\bar{\pi} = \mathrm{Ind}(\overline{G_{D\varepsilon}(F)}, \overline{G_D(F)}, \bar{\rho}_i)$.

This is proven in the same way as (2.43) via Clifford–Mackey theory (see also [LL, §2] and [GPS2, §1]), but it may also be a corollary of (2.43), using the "transfers"

$$\Pi(\overline{G_D(F)})_{\mathrm{gen}} \to \Pi(\overline{G(F)})_{\mathrm{gen}},$$

and

$$\Pi(\overline{G_{D_\varepsilon}(F)})_{\mathrm{gen}} \to \Pi(\overline{G_\varepsilon(F)})_{\mathrm{gen}},$$

obtained via the Selberg trace formula [DKV], [FK]. At any rate, these two facts (2.43–44) are the desired analogs of [Kaz, Lemma 1(a), p. 216].

For each character ω of $\overline{Z(F)^n}$, let $C_c(\bar{G}, \omega)_{\mathrm{gen}}$ denote the space of locally constant, compactly supported, genuine functions on $\overline{G(F)}$ transforming by ω on $\overline{Z(F)^n}$. Similarly, for G replaced by G_D. The analogs of [Kaz, Lemma 1(b), p. 216] are the following.

LEMMA (2.45). *Fix an unramified character ω of $\overline{Z(F)^r}$. For $\bar{\rho} \in \Pi(\overline{G_\varepsilon(F)})_{\mathrm{gen}}$, $\bar{\pi} \in \Pi^\varepsilon(\overline{G)F)})_{\mathrm{gen}}$ having central character ω, the following are equivalent*:

(a) $\bar{\pi} = \mathrm{Ind}(\overline{G_\varepsilon(F)}, \overline{G(F)}, \bar{\rho})$;

(b) *for some $c \in \mathbf{C}^\times$ and all $\bar{f} \in C_c(\bar{G}, \omega)_{\mathrm{gen}}$, the following identity of distributions holds*:

$$\mathrm{tr}(\bar{\pi}(\bar{f}) \circ A_\pi) = c \cdot \sum_{g \in \overline{G(F)}/\overline{G_\varepsilon(F)}} \varepsilon(g) \mathrm{tr}\, \bar{\rho}^g(\bar{f}\,|_{\overline{G_\varepsilon(F)}}),$$

where A_π intertwines $\bar{\pi}$ and $\varepsilon \otimes \bar{\pi}$ (the constant c depends on the choice of A_π).

The proof of this is given below. First, however, let me state the division algebra analog. Recall that one may identity the center of G_D with that of G, hence the center of $\overline{G_D(F)}$ may be identified with that of $\overline{G(F)}$.

LEMMA (2.46). *The analog of (2.45) holds if one replaces G by G_D throughout.*

At this point, perhaps it is worth observing that, for $\bar{\rho} \in \Pi(\overline{G_\varepsilon(F)})_{\text{gen}}$, the induced representation $\bar{\pi} := \text{Ind}(\overline{G_\varepsilon(F)}, \overline{G(F)}, \bar{\rho})$ *necessarily* satisfies $\bar{\pi} = \varepsilon \otimes \bar{\pi}$, since

$$\text{Ind}(\overline{G_\varepsilon(F)}, \overline{G(F)}, \varepsilon \otimes \bar{\rho}) = \varepsilon \otimes \text{Ind}(\overline{G_\varepsilon(F)}, \overline{G(F)}, \bar{\rho}),$$

and $\varepsilon \mid_{\overline{G_\varepsilon(F)}} \equiv 1$.

Now let me prove (2.45a) implies (2.45b) (the proof of the converse is omitted; it won't be used anyway). Write

$$\bar{\pi} \mid_{\overline{G_\varepsilon(F)}} = \bigoplus_{i=1}^{r} \bar{\rho}_i,$$

let W_i denote the space of $\bar{\rho}_i$, and let W denote the space of $\bar{\pi}$, so that $W = \oplus W_i$. Since $A_{\bar{\pi}}$ commutes with $\bar{\rho}_i$, Schur's lemma implies that $A_{\bar{\pi}} \mid_{W_i}$ acts like a scalar:

$$A_{\bar{\pi}} \mid_{W_i} = \lambda_i \cdot \text{id},$$

for $1 \leqq i \leqq r$. For $\bar{f} \in C_c(\bar{G})_{\text{gen}}$, then and (2.43) implies

$$\begin{aligned} \text{tr}\,(\bar{\pi}(\bar{f}) \circ A_{\bar{\pi}}) &= \sum_{i=1}^{r} \lambda_i \,\text{tr}\, \bar{\rho}_i(\bar{f} \mid_{\overline{G_\varepsilon(F)}}) \\ &= \lambda_1 \sum_{i=1}^{r} \zeta_r^{i-1} \,\text{tr}\, \bar{\rho}_i(\bar{f} \mid_{\overline{G_\varepsilon(F)}}), \end{aligned}$$

where ζ_r denotes a primitive rth root of unity. Here we used the fact that $\overline{G(F)}/\overline{G_\varepsilon(F)}$ acts transitively on the $\{W_i\}$, sending λ_i to $\varepsilon(g)\lambda_i$. This verifies that the desired identity holds.

The proof of (2.46) is similar and omitted.

Section 3

The Global Correspondence

The goal of this section is to prove a global "transfer" or correspondence of automorphic forms on $\overline{T(\mathbf{A}_K)}$ to automorphic forms on $\overline{G_D(\mathbf{A}_K)}$ (which may then be transferred to $\overline{GL(r, \mathbb{A}_K)}$ via Theorem 3.34 below). This yields a sort of metaplectic "endoscopic" transfer, borrowing some terminology of Langlands.

It is necessary to first define the global "Hecke algebras" for $\overline{T(\mathbb{A}_K)}$ and $\overline{G_D(\mathbb{A}_K)}$, following Kazhdan (see [Hen, §6]). This involves characterizing the ε-orbital integral in the style of Satake [Sat] and Vignéras [Vig1].

For a sketch of the proof of the global transfer in the non-metaplectic case, see [Hen, §§6.1–6.4]. Another method in the non-metaplectic case, yielding a more general result, is given in Arthur–Clozel [AC, ch. 3], where the transfer is an example of what they call "automorphic induction".

Throughout this section, we assume $(r, n) = 1$ without further mention.

§3.1. The global Hecke algebras

Let us retain the notation of §2.6. Let $T(\mathbb{A}_K)$ denote the CSG of $G_D(\mathbb{A}_K)$ defined by L/K. Following [Hen, §6.6, pp. 184–186], one can analogously define the metaplectic global Hecke algebras, denoted $\tilde{V}$ and $\tilde{V}^T$, on $\overline{T(\mathbb{A}_K)}$ and $\overline{G_D(\mathbb{A}_K)}$. The definition of $\tilde{V}^T$ largely depends on the characterization of metaplectic orbital integrals due to Vignéras [Vig2] (see also [KP2], [Hen, §5], [Vig1]). Later it will be convenient to consider the space of ε-orbital integrals as an intrinsically defined space of functions on $\overline{T(\mathbb{A}_K)}$, so actually it is important that this definition of $\tilde{V}^T$ is the "correct" one. The verification of this is sketched below.

Let me begin by introducing some more notation. Let ω denote a character of $\overline{Z(F)^n}$ and let $C_c(\overline{G_D(F)}, \omega)_{\text{gen}}$ denote the space of genuine, locally constant, compactly supported functions on $\overline{G_D(F)}$, transforming by ω^{-1} on $\overline{Z(F)^n}$. Let $C_c(\overline{G_D(F)})^{\text{reg}}, \omega)_{\text{gen}}$ denote the subspace of functions supported on $\overline{G_D(F)}^{\text{reg}}$, where

$$G_D(F)^{\text{reg}} := \{g \in G_D(F) \mid g \text{ generates an extension of } F \text{ of degree } r\},$$

and $\overline{G_D(F)}^{\text{reg}}$ denotes its lift to $\overline{G_D(F)}$. Let $T(F)^{\text{reg}} := T(F) \cap G_D(F)^{\text{reg}}$ and similarly for $\overline{T(F)}^{\text{reg}}$. One may define $C_c(\overline{T(F)}, \omega)_{\text{gen}}$ and $C_c(\overline{T(F)}^{\text{reg}}, \omega)_{\text{gen}}$ similarly. In order to introduce the Hecke algebras, it is convenient to introduce the following set. Let

S_D = a finite set of places of K containing S, $\Sigma_\infty(K)$, and containing all places where L/K is ramified, and which has the property that $\mathcal{O}_{K,S_D}$ is a principal ideal domain.

Here $\mathcal{O}_{K,S_D}$ denotes the S_D-integers of K. Let us choose a model of our algebraic group G_D/K which is defined over $\mathcal{O}_{K,S_D}$, denote it again by G_D, and let us fix an isomorphism defined over $\mathcal{O}_{K,S_D}$,

$$\phi : G_D \to G. \tag{3.1}$$

Let

$$\tilde{\theta} : \overline{Z(\mathbf{A})^n}/\overline{Z(K)^n} \to \mathbf{C} \tag{3.2}$$

denote a character; in case $n = 1$ denote this simply by θ, and let us take $\tilde{\theta}$ and θ to be "matching" in the sense that

$$\tilde{\theta}(z^n) = \theta(z), \tag{3.3}$$

for all $z \in Z(F)$. Finally, let $w \notin S_D$ be a place such that ε_w and $\tilde{\theta}_w \mid_{\overline{Z(K_w)^n}}$ are both unramified, and fix an arbitrarily large but finite set S_D^w containing $S_D \cup \{w\}$.

For each $v \in \Sigma_f(K)$, the local Hecke algebras $\mathscr{H}(\overline{G_D(K_v)}, \tilde{\theta}_v)_{\text{gen}}$, $\mathscr{H}(\overline{G(K_v)}, \tilde{\theta}_v)_{\text{gen}}$, and $\mathscr{H}(\overline{T(K_v)}, \tilde{\theta}_v)_{\text{gen}}$ have been defined already. For $\tilde{\theta}$ and θ satisfying (3.3), pulling back via the maps (3.1), (2.42), there is the following commutative diagram:

$$\begin{array}{ccc} \mathscr{H}(G(K_v), \theta_v) & \xrightarrow{\sim} & \mathscr{H}(G_D(K_v), \theta_v) \\ \downarrow & & \downarrow \\ \mathscr{H}(\overline{G(K_v)}, \tilde{\theta}_v)_{\text{gen}} & \xrightarrow{\sim} & \mathscr{H}(\overline{G_D(K_v)}, \tilde{\theta}_v)_{\text{gen}}, \end{array} \tag{3.4}$$

and also the natural map (via this diagram and (2.13))

$$\mathscr{H}(T(K_v), \theta_v) \to \mathscr{H}(\overline{T(K_v)}, \tilde{\theta}_v)_{\text{gen}}. \tag{3.5}$$

Note that the Weyl group $W_v := \text{Aut}\, T(K_v)$ acts on $\mathscr{H}(\overline{T(K_v)}, \tilde{\theta}_v)_{\text{gen}}$ via its action on $\mathscr{H}(T(K_v), \theta_v)$ using (3.5).

Now it is possible to define the *global metaplectic Hecke algebra* for $\overline{G_D(\mathbf{A}_K)}$. It is the space $\tilde{V} = \tilde{V}(\tilde{\theta})$ of all functions $\tilde{f} : \overline{G_D(\mathbf{A}_K)} \to \mathbf{C}$ such that

(3.6a) $\tilde{f} = \otimes_{v \in \Sigma(K)} \tilde{f}_v$;

(3.6b) for $v \notin S_D^{\omega}$, $\tilde{f}_v \in \mathscr{H}(\overline{G_D(K_v)}, \tilde{\theta}_v)_{\text{gen}} \cong \mathscr{H}(\overline{G(K_v)}, \tilde{\theta}_v)_{\text{gen}}$, satisfying $\tilde{f}_v = \tilde{f}_v^0$ for almost all $v \notin S_D^w$;

(3.6c) $\tilde{f}_w \in C_c(\overline{G_D(K_w)}, \tilde{\theta}_w)_{\text{gen}}$;

(3.6d) for $v \in S_D^w - (S \cup \omega)$, $\tilde{f}_v \in C_c(\overline{G_D(K_v)^{\text{reg}}}, \tilde{\theta}_v)_{\text{gen}} \cong C_c(\overline{G(K_v)^{\text{reg}}}, \tilde{\theta}_v)_{\text{gen}}$;

(3.6e) for $v \in S$, $\tilde{f}_v \in C_c(\overline{G_D(K_v)}, \tilde{\theta}_v)_{\text{gen}}$.

Before turning our attention to the Hecke algebra for $\overline{T(\mathbf{A}_K)}$, let me fix some

Haar measures. Choose the Tamagawa measure on both $G_D(\mathbf{A}_K)/Z(\mathbf{A}_K)$ and $\overline{G_D(\mathbf{A}_K)}/\overline{Z(\mathbf{A}_K)^n}$, and choose a decomposition

$$dg = \otimes_v dg_v, \quad d\bar{g} = \otimes_v d\bar{g}_v,$$

such that, for $v \notin S_D$,

$$\text{(3.7)} \qquad \mathrm{vol}[Z(K_v)G_D(\mathcal{O}_{K_v})/Z(K_v)] = \mathrm{vol}[\overline{Z(K_v)'G_D(\mathcal{O}_{K_v})}/\overline{Z(K_v)^n}] = 1.$$

Choose the Tamagawa measure on $T(\mathbf{A}_K)/Z(\mathbf{A}_K)$ and $\overline{T(\mathbf{A}_K)}/\overline{Z(\mathbf{A}_K)^n}$, and choose a decomposition

$$dt = \bigotimes_v dt_v, \quad d\bar{t} = \bigotimes_v d\bar{t}_v,$$

such that, for $v \notin S_D$, the quotients

$$Z(K_v)(T(K_v) \cap G_D(\mathcal{O}_{K_v}))/Z(K_v)$$

and

$$\overline{Z(K_v)^n}(\overline{T(K_v)} \cap \overline{G_D(\mathcal{O}_{K_v})})/\overline{Z(K_v)^n}$$

both get volume 1. For typographical simplicity, sometimes $d\bar{g}$ is abbreviated to dg, etc.

Define the *ε-orbital integeral* at $v \in \Sigma(K)$ by

$$\text{(3.8)} \qquad I(t, \tilde{f}_v) := \int_{\overline{G_D(K_v)}/\overline{Z(K_v)^n}} \varepsilon_v(g)\tilde{f}_v(g^{-1}tg)dg_v,$$

for $\tilde{f}_v \in C_c(\overline{G_D(K_v)})_{\mathrm{gen}}$, and define the *normalized ε-orbital integral* by

$$\text{(3.9)} \qquad \tilde{f}_v^{\,T}(t) := \Delta_v(p(t))I(t, \tilde{f}_v),$$

for all $t \in \overline{T(K_v)}^{\mathrm{reg}}$. The characterization of the image of the map $\tilde{f} \mapsto \tilde{f}^T$ is incorporated in the following notion. Define the *metaplectic global Hecke algebra* for $\overline{T(\mathbf{A}_K)}$ to be the space $\tilde{V}^T = \tilde{V}(\tilde{\theta})^T$ of all $\tilde{h} : \overline{T(\mathbf{A}_K)}^{\mathrm{reg}} \to \mathbf{C}$ such that

(3.10a) $\tilde{h} = \otimes_v \tilde{h}_v$;

(3.10b) for all $v \in \Sigma(K)$, $\tilde{h}_v = \tilde{f}_v^{\,T}$, for some $\tilde{f} \in \tilde{V}$;

(3.10c) for $v \notin S_D^w$, $\tilde{h}_v = \tilde{g}_v \,|_{\overline{T(K_v)}^{\mathrm{reg}}}$, for a unique $\tilde{g}_v \in \mathscr{H}(\overline{T(K_v)}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\mathrm{gen}}$, and $\tilde{f}_v = \tilde{f}_v^{\,0}$ for almost all such v; if $\tilde{f}_v = \tilde{f}_v^{\,0}$ then $\tilde{g}_v = \mathrm{id} \in \mathscr{H}(\overline{T(K_v)}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\mathrm{gen}}$.

The set of all such $\tilde{g}_v$ arising in this manner is the subspace

$$\mathscr{H}(\overline{T(K_v)}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\mathrm{gen}}^{W_v};$$

(3.10d) $\tilde{f}_w = \tilde{g}_w \mid_{\overline{T(K_v)}^{\text{reg}}}$, for a unique $\tilde{g}_w \in C_c(\overline{T(K_w)}, \tilde{\theta}_w \varepsilon_w^{r(r-1)/2})_{\text{gen}}$, and this $\tilde{g}_w$ is W_w-invariant. Furthermore, using [FK], one can choose $\tilde{f}_w$ such that $\tilde{f}_w^{\vee}$ is a matrix coefficient of a supercuspidal $\tilde{\pi}_w \in \Pi(\overline{G(K_w)})_{\text{gen}}$ such that $\varepsilon_w \otimes \tilde{\pi}_w \cong \tilde{\pi}_w$, the central character of $\tilde{\pi}_w$ is $\tilde{\theta}_w$, and such that $\tilde{f}_w^{\tilde{T}} \neq 0, \tilde{f}_w(1) = 1$;

(3.10e) for $v \in S_D^w - (S \cup \{w\})$, $\tilde{h}_v \in C_c(\overline{T(K_v)}^{\text{reg}}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\text{gen}}$, the set of such $\tilde{f}_v^{\tilde{T}} = \tilde{h}_v$ is the subspace

$$C_c(\overline{T(K_v)}^{\text{reg}}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\text{gen}}^{W_v};$$

(3.10f) for $v \in S$, $\tilde{h}_v \in C_c(\overline{T(K_v)}^{\text{reg}}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\text{gen}}$ and the set of $\tilde{h}_v$ is the subspace

$$\{\tilde{h}_v \in C_c(\overline{T(K_v)}^{\text{reg}}, \tilde{\theta}_v \varepsilon_v^{r(r-1)/2})_{\text{gen}} \mid \tilde{h}_v(\sigma t) = \varepsilon_v(r_{K_v}(\sigma)^{-a_v})\tilde{h}_v(t), \text{ for all } \sigma \in \operatorname{Gal}(L_v/K_v)\},$$

where $\sigma t = (\sigma p(t), \zeta)$, for $t := (p(t), \zeta)$, and a_v/r is the invariant of D_v (thus a_v is well-defined mod r, see Weil [W, p. 224], for example).

With this definition of $\tilde{V}^{\tilde{T}}$, one may now consider the normalized ε-orbital integral $\tilde{f}^{\tilde{T}}$, for each $\tilde{f} \in \tilde{V}$, as a function on $\overline{T(\mathbf{A}_K)}$ which transforms on $\overline{Z(\mathbf{A}_K)^r}$ by $\tilde{\theta}^{-1}\varepsilon^{-r(r-1)/2}$. This function $\tilde{f}^{\tilde{T}}$ on $\overline{T(\mathbf{A}_K)}$ satisfies a very important identity (for us anyway!) which will be given in the next subsection.

The charactrization of the $\tilde{f}^{\tilde{T}}$ implicit in this definition must be verified. Let me begin with some elementary lemmas. They are simply the "obvious" metaplectic analogs of the corrresponding results in [Hen, §§5.17–5.22], so I will merely state them without proof. The first few of the following results use the notation of §2.4. In particular, the notion of an ε-orbital integral is not the same as in (3.8).

LEMMA (3.11). *The map*

$$\mathscr{H}(\bar{G})_{\text{gen}} \to \mathscr{H}(\bar{T})_{\text{gen}}$$

defined by $\tilde{f} \mapsto \tilde{S}_H \circ \tilde{j}_*(\tilde{f})$ *is dual to* ${}^LT \hookrightarrow {}^LG^0$. *Under this map, the image of* $\mathscr{H}(\bar{G})_{\text{gen}}$ *is* $\mathscr{H}(\bar{T})_{\text{gen}}^W$, *with* $W = \operatorname{Aut}_F T$. *Similarly, the image of*

$$\mathscr{H}(\bar{G},\omega)_{\text{gen}} \to \mathscr{H}(\bar{T}, \omega\varepsilon^{r(r-1)/2})_{\text{gen}},$$

defined by $\tilde{f} \mapsto \tilde{f}^{\tilde{T}}$, *is the subspace* $\mathscr{H}(\bar{T}, \omega\varepsilon^{r(r-1)/2})_{\text{gen}}^W$.

REMARK. For this, see (2.20) in §2 above and [Hen, p. 175].

LEMMA (3.12). *Take* $\tilde{f}\in C_c(\tilde{G},\omega)_{\text{gen}}$. *There is a unique* $\tilde{h}\in C_c(\tilde{T},\omega\varepsilon^{r(r-1)/2})^W_{\text{gen}}$ *such that*

$$\tilde{h}(t)=\Delta(p(t))I(t,\tilde{f}),$$

for all $t\in\overline{T(F)}^{\text{reg}}$.

REMARK. For the case $n=1$, see [Hen, Proposition 5.18, p. 174] and [Kaz, Proposition 3.2, p. 231].

Now we take E/F to be an arbitrary extension of degree m, possibly ramified, $T(F)$ to be a maximal torus in $G(F)$, and ε to be a character of $F^\times$ trivial on $\det(T(F))$, possibly ramified. For each $w\in W$, pick $x_w\in G(F)$ such that

$$x_w t x_w^{-1}=t^w, \tag{3.13}$$

for all $t\in T(F)$.

LEMMA (3.14). *For* $\tilde{f}\in C_c(\tilde{G},\omega)_{\text{gen}}$, *the function* $t\mapsto I(t,\tilde{f})$ *belongs to* $C_c(\overline{T(F)}^{\text{reg}},\omega)_{\text{gen}}$ *and satisfies, for all* $\omega\in W$, $t\in\overline{T(F)}$,

$$I(t^\omega,\tilde{f})=\varepsilon(x_\omega)I(t,\tilde{f}).$$

Conversely, any function in $C_c(\overline{T(F)}^{\text{reg}},\omega)_{\text{gen}}$ *satisfying this property is of the form* $t\mapsto I(t,\tilde{f})$ *for some* $\tilde{f}\in C_c(\tilde{G}^{\text{reg}},\omega)_{\text{gen}}$.

REMARK. There is no "$\varepsilon^{r(r-1)/2}$" factor since the Δ-factor isn't present. The case $n=1$ in [Hen, Proposition 5.21, p. 177].

There is a division algebra analog of (3.14) which is also needed. For this, define the *ε-orbital integral* by

$$I(t,\tilde{f}):=\int_{\overline{G_D(F)}/\overline{T(F)}}\varepsilon(x)\tilde{f}(x^{-1}tx)dx, \tag{3.15}$$

for $\tilde{f}\in C_c(\overline{G_D(F)})_{\text{gen}}$. Let $T(F)$ denote a maximal torus in $G_D(F)$ corresponding to a cyclic extension E of F in our division algebra D. Let ε denote a character of $F^\times$ defining E.

LEMMA (3.16). *For* $\tilde{f}\in C_c(\overline{G_D(F)},\omega)_{\text{gen}}$ *the function* $t\mapsto I(t,\tilde{f})$ *belongs to* $C_c(\overline{T(F)}^{\text{reg}},\omega)_{\text{gen}}$ *and satisfies, for* $w\in W$, $t\in\overline{T(F)}^{\text{reg}}$, *the identity*

$$I(t^w,\tilde{f})=\varepsilon(x_w)I(t,\tilde{f}),$$

where x_w is defined as in (3.13). *Conversely, any function in* $C_c(\overline{T(F)}^{\mathrm{reg}}, \omega)_{\mathrm{gen}}$ *satisfying this property is of the form* $t \to I(t, \tilde{f})$, *for some* $\tilde{f} \in C_c(\overline{G_D(F)}^{\mathrm{reg}}, \omega)_{\mathrm{gen}}$.

REMARK. The case $n = 1$ of this is proven in [Hen, pp. 178–179].

Now the verification of the characterization in (3.10) of the functions of the form $\tilde{f}^{\tilde{T}}$ proceeds exactly as in Henniart's proof of his Lemma 6.6 [Hen, pp. 184–186]. As the details are not very illuminating, they will be omitted.

§3.2. Automorphic forms on $\overline{G_D(\mathbf{A}_K)}$ and $\overline{T(\mathbf{A}_K)}$

We begin by introducing a little more notation, while preserving that of §3.1. Let $L^2(\overline{G_D(\mathbf{A}_K)}, \tilde{\theta})$ denote the space of $\tilde{f}$: $\overline{G_D(\mathbf{A}_K)}/G_D(K) \to \mathbf{C}$ such that

(3.17a) $\tilde{f}(gz) = \tilde{\theta}(z)\tilde{f}(g)$, for all $z \in \overline{Z(\mathbf{A}_K)^n}$,

(3.17b) $\tilde{f}$ is genuine, and

(3.17c) $\int_{\overline{G_D(\mathbf{A}_K)}/\overline{Z(\mathbf{A}_K)^n}G(D(K))} |\tilde{f}(g)|^2 dg < \infty$.

Recall that $\tilde{\theta}$ is assumed to be unitary. Such a function $\tilde{f}$ might be referred to as a *square-integrable automorphic form* on $\overline{G_D(\mathbf{A}_K)}$. Let $\tilde{R}_D$ denote the right regular representation of $\overline{G_D(\mathbf{A}_K)}$ on $L^2(\overline{G_D(\mathbf{A}_K)}, \tilde{\theta})$, and write its decomposition into irreducibles as

$$\tilde{R}_D = \bigoplus_{\tilde{\pi}} m(\tilde{\pi})\tilde{\pi}, \qquad \tilde{\pi} = \bigotimes_{v} \tilde{\pi}_v, \tag{3.18}$$

where $\tilde{\pi}_v \in \Pi(\overline{G_D(K_v)})_{\mathrm{gen}}$. denote by ε the multiplication operator

$$\tilde{f}(g) \mapsto \varepsilon(g)\tilde{f}(g),$$

for $\tilde{f} \in L^2(\overline{G_D(\mathbf{A}_K)}, \tilde{\theta})$. This operator permutes the components (not transitively though!) of $\tilde{R}_D$; of course, a component fixed by ε satisfies $\tilde{\pi} \cong \varepsilon \otimes \tilde{\pi}$. Let $L^2(\overline{T(\mathbf{A}_K)}, \tilde{\theta}\varepsilon^{r(r-1)/2})$ denote the space of functions $\tilde{h}: \overline{T(\mathbf{A}_K)}/T(K) \to \mathbf{C}$ such that

(3.19a) $\tilde{h}(zt) = \tilde{\theta}(z)\varepsilon(p(z))^{r(r-1)/2}\tilde{h}(t)$, for all $z \in \overline{Z(\mathbf{A}_K)^n}$,

(3.19b) $\tilde{h}$ is genuine, and

(3.19c) $\int_{\overline{T(\mathbf{A}_K)}/T(K)\overline{Z(\mathbf{A}_K)^n}} |\tilde{h}(t)|^2 dt < \infty$.

This might be referred to as the space of *automorphic forms* on $\overline{T(\mathbf{A}_K)}$. Let us write the decomposition into irreducibles of the right regular representation $\tilde{R}^{\tilde{T}}$ of $\overline{T(\mathbf{A}_K)}$ on this space as

$$\tilde{R}^{\tilde{T}} = \bigoplus_{\chi} = m(\chi)\chi, \qquad \chi = \bigotimes_{v} \chi_v, \tag{3.20}$$

where $\chi_v \in \Pi(\overline{T(K_v)})_{\text{gen}}$.

As a matter of convention, if $n = 1$ then write f in place of $\tilde{f}$, R_D in place of $\tilde{R}_D$, etc.

Let me state a simple observation.

FACT (3.21). For $\tilde{f} \in \tilde{V}$, the operator $\tilde{R}_D$ is of trace class and

$$\operatorname{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon) = \sum_{\gamma \in G_D(K)/Z(K)^n} I_\gamma^\varepsilon(\tilde{f}),$$

where

$$I_\gamma^\varepsilon(\tilde{f}) := \int_{\overline{G_D(\mathbf{A}_K)}/\overline{Z(\mathbf{A})}^n \overline{G_D(K)}_\gamma} \varepsilon(x) \tilde{f}(x^{-1}\gamma x)dx.$$

(Recall that $\overline{G_D(\mathbf{A}_K)}$ splits over $G_D(K)$, so the centralizer also must split: $\overline{G_D(K)}_\gamma = G_D(K)_{p(\gamma)} \times \mu_n$.)

PROOF. As in the non-metaplectic case, one simply notes that the trace above is given by the ε-twisted integral of the kernel function along the diagonal:

$$(3.22) \qquad (\tilde{R}_D(\tilde{f}) \circ \varepsilon)(F)(x) = \int_{\overline{G_D(\mathbf{A}_K)}/\overline{Z(\mathbf{A})}^n G_D(K)} K_{\tilde{f}}(x, y)\varepsilon(y)F(y)dy,$$

where

$$K_{\tilde{f}}(x, y) := \sum_{\gamma \in G_D(K)/Z(K)^n} \tilde{f}(x^{-1}\gamma y),$$

and

$$(3.23) \qquad \operatorname{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon) = \int_{\overline{G_D(\mathbf{A}_K)}/\overline{Z(\mathbf{A})}^n G_D(K)} K_{\tilde{f}}(x, x)\varepsilon(x)dx.$$

Below, it's shown that, for $\tilde{f} \in \tilde{V}$, "most" of the global ε-orbital integrals $I_\gamma^\varepsilon(\tilde{f})$ vanish in the sum in (3.21). In fact, unless $\gamma \in \overline{G_D(K)}$ is $\overline{G_D(\mathbf{A}_K)}$-conjugate to an element in $\overline{G_D(\mathbf{A}_K)}^{\text{reg}}$, one has $I_\gamma^\varepsilon(\tilde{f}) = 0$ (remember that the "S-part" of $\tilde{f}$ is supported on the "S-part" of $\overline{G_D(\mathbf{A}_K)}^{\text{reg}}$). In the case $n = 1$, what happens is that $I_\gamma^\varepsilon(f) = 0$ unless γ is in fact $G_D(\mathbf{A}_K)$-conjugate to an element in $T(\mathbf{A}_K)^{\text{reg}}$, where T is the torus associated to the cyclic extension L/K corresponding to ε; this allows one to write ([Kaz, Proposition 4.1])

$$\operatorname{tr}(R_D(f) \circ \varepsilon) = \sum_{t \in T(K)^{\text{reg}}/Z(K)} \int_{G_D(\mathbf{A}_K)/T(\mathbf{A}_K)} \varepsilon(x) f(x^{-1}(x)dx,$$

for $f \in V$. The expression relevant for us is, of course, its metaplectic analog. I claim that the metaplectic analog of this is the "obvious" one, namely, that the following hold.

LEMMA (3.24). *If $I_\gamma^\varepsilon \not\equiv 0$ (as a distribution on $\tilde{V}$) then γ is $\overline{G_D(\mathbf{A}_K)}$-conjugate to an element in $\overline{T(\mathbf{A}_K)}^{\mathrm{reg}}$.*

REMARK. From Flicker–Kazhdan theory, ones knows that, in fact, in order for the ε-orbital integral to be nonzero, γ must be conjugate to an nth power of a regular element of $\overline{T(\mathbf{A}_K)}$. The point is that the centralizer in $\overline{G_D(\mathbf{A}_K)}$ of a regular element in $T(K)$ is $\overline{T(\mathbf{A}_K)}$, hence belongs to $\overline{G_{D_\varepsilon}(\mathbf{A}_K)}$; we may therefore apply the formula

$$\int_{\overline{G_D(\mathbf{A}_K)}/\overline{T(\mathbf{A}_K)}} \varepsilon(x)\tilde{f}(x^{-1}tx)dx = \sum_{g \in \overline{G_D(\mathbf{A}_K)}/\overline{G_{D_\varepsilon}(\mathbf{A}_K)}} \varepsilon(g) \int_{\overline{G_{D_\varepsilon}(\mathbf{A}_K)}} \tilde{f}(x^{-1}g^{-1}tgx)dx. \tag{3.25}$$

PROOF. The argument follows Kazhdan's argument fairly closely, so only the necessary (slight) modification will be sketched. Let $\mathrm{pr}: \mathbf{A}_K^\times \to \mathbf{A}_K^\times/K^\times$ denote the projection onto the idele class group. First, I claim that $I_\gamma^\varepsilon \equiv 0$ if

$$\mathrm{pr} \circ \mathrm{Nrd} \circ p(\overline{G_D(\mathbf{A}_K)}_\gamma) \not\subseteq \mathrm{pr} \circ N_{L/K}(\mathbf{A}_L^\times).$$

Indeed, if there is a $y \in \overline{G_D(\mathbf{A}_K)}$ with $\gamma = y^{-1}\gamma y$ and such that $\mathrm{pr}(\mathrm{Nrd}\, p(y))$ does not belong to $\mathrm{pr}(N_{L/K}(\mathbf{A}_L^\times))$ then $\varepsilon(y) \neq 1$. On the other hand,

$$I_\gamma^\varepsilon(\tilde{f}) = I_{y^{-1}\gamma y}^\varepsilon(\tilde{f}) = \varepsilon(y) I_\gamma^\varepsilon(\tilde{f}),$$

so $I_\gamma^\varepsilon(\tilde{f}) = 0$, as claimed. The remainder of the argument follows [Kaz, p. 235]. □

Combining (3.21) and (3.24) yields the following

LEMMA (3.26). *For $\tilde{f} \in \tilde{V}$, we have*

$$\mathrm{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon) = \sum_{\gamma \in (T(K)^n)^{\mathrm{reg}}/Z(K)^n} I_\gamma^\varepsilon(\tilde{f}).$$

Note that, via the Selberg trace formulas for $\overline{T(\mathbf{A}_K)}$ and $\overline{T(\mathbf{A}_K)^n}$, there is a natural 1–1 correspondence between the irreducible automorphic representations of $\overline{T(\mathbf{A}_K)}$ and the characters of $\overline{T(\mathbf{A}_K)^n}$. For reasons stated in the introduction, we shall work with $\overline{T(\mathbf{A}_K)}$ as much as possible.

This and lemma (3.26), when combined with the Selberg trace formula for $\overline{T(\mathbf{A}_K)^n}$ (i.e., Poisson summation), yields the following

PROPOSITION (3.27). *For all $\tilde{f} \in \tilde{V}$, we have*

$$\operatorname{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon) = \operatorname{tr}(\tilde{R}^T(\tilde{f})).$$

REMARK. Our next task is to decompose this identity into linearly independent pieces, following [Hen, §6].

The space $L^2(\overline{T(\mathbf{A}_K)}, \tilde{\theta}\varepsilon^{r(r-1)/2})$ decomposes into countably many irreducible subspaces W_χ, which are infinite-dimensional if $n > 1$ and one-dimensional if $n = 1$. Here χ denotes an automorphic representation of $\overline{T(\mathbf{A}_K)}$, in the terminology of (3.18). Let

$$\Pi(\overline{T(\mathbf{A}_K)})_{\mathrm{gen}}$$

denote the set of equivalence classes of such χ, so

$$L^2(\overline{T(\mathbf{A}_K)}, \tilde{\theta}\varepsilon^{r(r-1)/2}) = \bigoplus_{\chi \in \Pi(\overline{T(\mathbf{A}_K)})_{\mathrm{gen}}} m(\chi) W_\chi. \tag{3.28}$$

Let

$$T_\chi(\tilde{h}) := \int_{\overline{T(\mathbf{A}_K)}/\overline{Z(\mathbf{A}_K)}^n} \operatorname{tr} \chi(t) \tilde{h}(t) dt, \tag{3.29}$$

for $\tilde{h} \in \tilde{V}^T$, so we have

$$\operatorname{tr}(\tilde{R}^T(\tilde{h})) = \sum_{\chi \in \Pi(\overline{T(\mathbf{A}_K)})_{\mathrm{gen}}} m(\chi) T_\chi(\tilde{h}). \tag{3.30}$$

The $T_\chi(\tilde{h})$ are not all linearly independent, so we have not yet succeeded in finding the "pieces" in the remark (3.27). Indeed, as mentioned in §1, it is a consequence of Clifford–Mackey theory that all the χ_v's are induced from characters of $\overline{T(K_v)^n}$; there is an action of $\mathrm{Gal}(L/K) \hookrightarrow \mathrm{Aut}(L_v)$ on these χ_v's and this gives an action of $\mathrm{Gal}(L/K)$ on the χ's. The fact that not all the T_χ's are linearly independent is basically due to the following

FACT (3.31). For all $\sigma \in \mathrm{Gal}(L/K)$, $T_\chi = T_{\chi^\sigma}$.

PROOF. Define x_σ by $t^\sigma = x_\sigma^{-1} t x_\sigma$, for $\sigma \in \mathrm{Gal}(L/K)$. From (3.29) we get

$$\begin{aligned} T_{\chi^\sigma}(\tilde{f}_T) &= \int_{\overline{T(\mathbf{A}_K)}/\overline{Z(\mathbf{A}_K)}^n} \operatorname{tr} \chi(t^\sigma) \tilde{f}^T(t) dt \\ &= \int_{\overline{T(\mathbf{A}_K)}/\overline{Z(\mathbf{A}_K)}^n} \operatorname{tr} \chi(t) \tilde{f}^T(x_\sigma t x_\sigma^{-1}) dt \\ &= \varepsilon(x_\sigma) T_\chi(\tilde{f}^T), \end{aligned}$$

by definition of the ε-orbital integral. □

The second property which the T_χ satisfy is the following

FACT (3.32). Let $\chi \in \Pi(\overline{T(\mathbf{A}_K)})_{\text{gen}}$ be such that, for some $v \in S$ and all $\sigma \in \text{Gal}(L/K)$, $\chi_v^\sigma = \chi_v$; then $T_\chi = T_{\chi^\sigma}$.

REMARK. The proof is the same as the case of $n = 1$; see [Kaz, p. 240].

The last property of the T_χ tells us exactly which of the non-zero T_χ's are linearly independent.

FACT (3.33). Let $\chi_1, \ldots, \chi_N \in \Pi(\overline{T(\mathbf{A}_K)})_{\text{gen}}$ be such that $T_{\chi_i} \not\equiv 0$ for $1 \leqq i \leqq N$ and such that $\chi_i^\sigma \neq \chi_j$, for all $1 \leqq i \neq j \leqq N$ and all $\sigma \in \text{Gal}(L/K)$; then the T_{χ_i} are linearly independent.

REMARK. For the proof in the case $n = 1$, see [Kaz, p. 241].

The linearly independent pieces of the right-hand side of (3.27) are the $\text{tr}\, \tilde{\pi}(\tilde{f})$ where $\tilde{\pi}$ is ε-invariant.

We need the following result, implicit in [KP2, §6]:

THEOREM (3.34) ("Kazhdan-Patterson lifting"). *Assume* $(n, r) = 1$ *and that K is totally imaginary. Then there is an injection*

$$\Pi(\overline{G_D(\mathbf{A}_K)})_{\text{gen}} \to \Pi(\overline{\text{GL}(r, \mathbf{A}_K)})_{\text{gen}},$$

consistent with (2.42).

REMARK. This result may be regarded as the metaplectic analog of the Deligne–Kazhdan lifting [DKV].

Since $|\text{Gal}(L/K)| = r$, clearly (3.31) gives

$$\sum_{\sigma \in \text{Gal}(L/K)} T_{\chi^\sigma} = rT_\chi,$$

so by (3.30) it follows that

$$\text{tr}(\tilde{R}^T(\tilde{h})) = r \cdot \sum_{\chi \in X_n(\tilde{\theta}, \varepsilon, D)} T_\chi(\tilde{h}), \tag{3.35}$$

for all $\tilde{h} \in \tilde{V}^T$, where χ runs over the set $X_n(\tilde{\theta}, \varepsilon, D)$ of all χ's such that $\chi_v^\sigma \neq \chi_v$, for all $v \in S$, counted according to their multiplicity. The identity (3.35) is quite basic to our argument; as we've already noted, according to (3.33), the distinct non-zero T_χ's are linearly independent. These distributions T_χ on $\tilde{V}^T$ may be lifted, via (3.10), to a unique distribution $\tilde{T}_\chi$ on $\tilde{V}$:

$$\tilde{T}_\chi(\tilde{f}) := T_\chi(\tilde{f}^T), \tag{3.36}$$

for $\tilde{f} \in \tilde{V}$. Now let us consider the distribution $\tilde{f} \mapsto \mathrm{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon)$. Let $Y'_n(\tilde{\theta}, \varepsilon)$ denote the set of isotypical components of $L^2(\overline{G_D(\mathbf{A}_K)}, \tilde{\theta})$ which are invariant under the multiplication operator ε. Put

$$T_{\tilde{\pi}}(\tilde{f}) := \mathrm{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon \mid \tilde{\pi}); \tag{3.37}$$

for $\tilde{\pi} \in Y'_n(\tilde{\theta}, \varepsilon)$ it easy to see that

$$\mathrm{tr}(\tilde{R}_D(\tilde{f}) \circ \varepsilon) = \sum_{\tilde{\pi} \in Y'_n(\tilde{\theta}, \varepsilon)} m(\tilde{\pi}) T_{\tilde{\pi}}(\tilde{f}), \tag{3.38}$$

for $\tilde{f} \in \tilde{V}$. The proposition (3.27) above, (3.35), and (3.38) yield the fundamental equality

$$\sum_{\tilde{\pi} \in Y'_n(\tilde{\theta}, \varepsilon)} m(\tilde{\pi}) T_{\tilde{\pi}}(\tilde{f}) = r \cdot \sum_{\chi \in X_n(\tilde{\theta}, \varepsilon, D)} \tilde{T}_\chi(\tilde{f}). \tag{3.39}$$

Let $Y_n(\tilde{\theta}, \varepsilon)$ denote the set of $\tilde{\pi} \in Y'_n(\tilde{\theta}, \varepsilon)$ such that

(3.40a) $\tilde{\pi}_v$ is supercuspidal for at least two places, at least one of them not in S,

(3.40b) at least at one of these two places, $\tilde{\pi}_v$ maps under the metaplectic (Flicker–Kazhdan) correspondence to a supercuspidal π_v,

(3.40c) $T_{\tilde{\pi}} \neq 0$ (as a distribution), and

(3.40d) $(n, N) = 1$ where N denotes the least common multiple of all the composite natural numbers less than or equal to r.

Thus for $\tilde{\pi} \in Y_n(\tilde{\theta}, \varepsilon)$, we have $m(\tilde{\pi}) = 1$. The set $Y_n(\tilde{\theta}, \varepsilon)$ is our (sub)set of linearly independent "pieces" of (3.38). The conditions (3.40) have been chosen in such a way that the elements of $Y_n(\tilde{\theta}, \varepsilon)$ satisfy the "strong multiplicity one" (rigidity) theorem and occur with multiplicity one (by [FK, §28 and Introduction] and the "Kazhdan–Patterson lifting" quoted above, the representations in $Y_n(\tilde{\theta}, \varepsilon)$ lift from $\Pi(\overline{G_D(\mathbf{A}_K)})_{\mathrm{gen}}$ to a subset of $\Pi(\overline{\mathrm{GL}(r, \mathbf{A}_K)})_{\mathrm{gen}}$ satisfying multiplicity one and rigidity; see also [Hen. §6.1–64]). Comparing the sets $Y_n(\tilde{\theta}, \varepsilon)$, $X_n(\tilde{\theta}, \varepsilon, D)$ using (3.39) and a simple combinatorial lemma (see [Kaz, Lemma 4.7]), one obtains a global (set-theoretic) transfer, stated below.

THEOREM (3.41). *Assume $(n, r) = 1$, $(N, n) = 1$. There is an injection $Y_n(\tilde{\theta}, \varepsilon) \hookrightarrow X_n(\tilde{\theta}, \varepsilon, D)$, denoted $\tilde{\pi} \mapsto \chi = \chi_{\tilde{\pi}}$, such that*

$$T_{\tilde{\pi}} = r\tilde{T}_{\chi}.$$

Furthermore, this transfer is consistent, for almost all v, with the local L-group correspondence

$$\Pi_{\text{unr.p.s.}}(\overline{G_D(K_v)})_{\text{gen}} \to \Pi(\overline{T(K_v)})_{\text{gen}}$$

(see §1 and §2).

REMARK. For the proof, see [Kaz, §4 (esp. p. 243)]. The supercuspidal condition in the definition of Y_n was added as a safety precaution (see [Hen, §§6.3, 6.4, 6.16]) and might be extraneous.

§3.3. A local character identity

In the previous subsection, we saw a correspondence between certain components $\tilde{\pi}$ of $L^2(\overline{G_D(\mathbf{A}_K)}, \tilde{\theta})$ and certain components χ of $L^2(\overline{T\mathbf{A}_K}), \tilde{\theta}\varepsilon^{r(r-1)/2})$. Here I want to indicate a corollary which follows from Kazhdan's Proposition 5.1 [Kaz] and the Weyl integration formula (in the form given, for example, in [F, p. 142]). If $\tilde{\pi}$ corresponds to χ then one can show that the local components satisfy

$$\chi^{\varepsilon}_{\tilde{\pi}_v}(t) = c_v \cdot \Delta(p(t))^{-1} \sum_{\sigma_v \in \mathrm{Gal}(L_v/K_v)} \mathrm{tr}(\chi_v^{\sigma_v})(t), \tag{3.42}$$

for $t \in \overline{T(K_v)}^{\text{reg}}$, where $c_v \neq 0$ is some constant, and $\chi^{\varepsilon}_{\tilde{\pi}_v}$ is the locally constant function on $\overline{G_D(K_v)}^{\text{reg}}$ associated to the distribution $\mathrm{tr}(\tilde{\pi}_v(\tilde{f}) \circ A_{\tilde{\pi}_v})$ in (2.45) (its existence is a consequence of a theorem of Harish-Chandra and Clozel [Cloz]). In other words, if $\tilde{\pi}_v = \mathrm{Ind}(\overline{G_{D_\varepsilon}(K_v)}, \overline{G_D(K_v)}, \tilde{\rho}_v)$ then one has

$$\sum_{g \in \overline{G_D(K_v)}/\overline{G_{D_\varepsilon}(K_v)}} \varepsilon(g)\chi_{\tilde{\rho}_v^g}(t) = c_v \cdot \Delta(p(t))^{-1} \sum_{\sigma_v \in \mathrm{Gal}(L_v/K_v)} \mathrm{tr}(\chi_v^{\sigma_v})(t), \tag{3.43}$$

where $\chi_{\tilde{\rho}_\theta}$ is the locally constant distribution on $\overline{G_{D\varepsilon}(K_v)}^{\text{reg}}$ associated to $\mathrm{tr}\,\tilde{\rho}_v(\tilde{f})$, for $\tilde{f}$ supported on $\overline{G_{D\varepsilon}(K_v)}$. In the case $r = 2$, there is a similarity between (3.43) and an identity of Labesse–Langlands [LL, (2.4)].

§3.4. Remarks on preservation of L-factors and ε-factors

We might as well assume that our covering groups are non-trivial, i.e. $n \geqq 2$, since for $n = 1$ the preservation of L-factors and ε-factors is due to Henniart [Hen]. Also, let me restrict the discussion to the "good" non-archimedian places. In this case, the L-factors have only been defined, to the best of my knowledge, for $r = 2$ and $n = 2$ (by Gelbart and Piatetski-Shapiro [GPS1]) or $r \geqq 2$, $n \geqq 2$ for the unramified finite places (by Bump and Hoffstein [BH, Introduction]). The ε-factors have only been defined for $r = 2$ and $n = 2$ (again, by Gelbart and Piatetski-Shapiro [GPS1]).

In [GPS1], it is shown that the Shimura correspondence, for $n = r = 2$, preserves L-factors and ε-factors (at least when $p > 2$). It seems clear that, for the unramified finite places, the metaplectic (Flicker–Kazhdan) correspondence [FK] preserves the Bump–Hoffstein L-factors.

Henniart [Hen] showed that Kazhdan's endoscopic lifting preserves L-factors and ε-factors. Using L-parameters as in §1 of this paper, it can be shown that the Langlands–Hecke L-factors, defined in the obvious way, are preserved under the local unramified correspondence

$$\Pi_{\mathrm{unr}}(\overline{T(F)})_{\mathrm{gen}} \to \Pi_{\mathrm{unr}}(T(F)),$$

and that (as we've already shown) the local diagram

$$\begin{array}{ccc} \Pi_{\mathrm{unr}}(\overline{T(F)})_{\mathrm{gen}} & \xrightarrow{\mathrm{Sh}} & \Pi_{\mathrm{unr}}(T(F)) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Kazhdan}} \\ \Pi_{\mathrm{unr}}(\overline{\mathrm{GL}(r, F)})_{\mathrm{gen}} & \xrightarrow{\mathrm{Sh}} & \Pi_{\mathrm{unr}}(\mathrm{GL}(r, F)) \end{array} \tag{3.44}$$

commutes. Putting all these facts together, it seems clear that the metaplectic Kazhdan lifting preserves L-factors at the unramified finite places, for $n \geqq 1$, $r \geqq 2$. It remains to consider the case $r = 2$, $n = 2$ and the ramified finite places.

Note first that the triangle

$$\begin{array}{ccc} & & \Pi(\overline{\mathrm{GL}(2, \mathbf{A}_K)})_{\mathrm{gen}} \\ & \nearrow & \big\downarrow{\scriptstyle \mathrm{Shimura}} \\ \Pi(T(\mathbf{A}_K)) & \longrightarrow & \Pi(\mathrm{GL}(2, \mathbf{A}_K)) \end{array} \tag{3.45}$$

commutes (it commutes at the unramified places, by (3.44), and one uses the "strong multiplicity one" (rigidity) theorem to obtain the global result). In (3.45), the downward arrow on the right is the correspondence of [GPS1] (for $n = 2$) or of [F] (for $n \geqq 2$), the bottom arrow is the lift of Labesse–Langlands [LL] (generalized later by Kazhdan), and the top arrow is the composition of

$$\Pi(T(\mathbf{A}_K)) \xrightarrow{\mathrm{Sh}^{-1}} \Pi(\overline{T(\mathbf{A}_K)})_{\mathrm{gen}}, \tag{3.46}$$

implicitly obtained above (see the remarks following Lemma 3.26), and the "metaplectic Kazhdan lifting",

$$\Pi(\overline{T(\mathbf{A}_K)})_{\text{gen}} \to \Pi(\overline{\mathrm{GL}(r, \mathbf{A}_K)})_{\text{gen}}. \tag{3.47}$$

We want to know if the top arrow in (3.45) preserves L-factors and ε-factors. They are preserved when $n = r = 2$ since, by Gelbart and Piatetski-Shapiro [GPS1] and Henniart [Hen], the other arrows preserve them.

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